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Continuity of Higher Derivations on Some Semiprime Banach Algebras

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ABSTRACT. In this paper, it is shown that automatic continuity of derivations on some semiprime Banach algebras can be extended to higher derivations. In particular, we show that if every prime ideal is closed in a commutative semiprime Banach algebra then every higher derivation on it is continuous.

I. Introduction

For m in N, a higher derivation of rank m (respectively infinite rank) from an algebra A into an algebra B is a sequence $\{F_0, \dots, F_m\}$ (resp. $\{F_0, F_1, \dots\}$ of linear operators from A into B satisfying

$$F_n(ab) = \sum_{i=0}^{n} F_i(a)F_{n-i}(b)$$

for each $n = 0, 1, \dots, m$ (resp. $n = 0, 1, \dots$) and all a, b, in A. A higher derivation of rank m (resp. infinite rank) is said to be continuous if F_n is continuous for each $n = 0, 1, \dots, m$ (resp. $n = 0, 1, 2, \dots$). It is said to be onto if F_0 maps A onto B. For definitions and elementary properties of Banach algebras we refer to [**2**].

R.J. Loy [**12**] obtained that the result of B.E. Johnson and A.M. Sinclair [**10**] giving the automatic continuity of derivations on semisimple Banach algebras can be extended to higher derivations whose domain algebra is the same as the range algebra and where F_0 is the identity map. Also N.P. Jewell [**8**] extended this result (i) by allowing the domain algebra to be any Banach algebra whatsoever, (ii) by allowing the range algebra to include a wider class than just semisimple algebra and (iii) by weakening the condition that F_0 be the identity map.

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R.V. Garimella [5] showed that if every prime ideal is closed in a commutative semisimple Banach algebra with unit then every derivation on it is continuous. Also if derivations are continuous on integral domains then they are continuous on semiprime Banach algebras. In this paper, it is shown that the result of Garimella can be extended to a higher derivation.

If S is a linear operator from a Banach space X into a Banach space Y, then the separating space $\mathfrak{S}(S)$ of S is defined by

$$\mathfrak{S}(S) = \{y \in Y : \text{ there are } x_n \to 0 \text{ with } Sx_n \to y\}$$

$\mathfrak{S}(S)$ gives us a measure of discontinuity of the linear operator S since the closed graph theorem shows that S is continuous if and only if $\mathfrak{S}(S) = (0)$ and the following Stability Lemma [9] for the separating space is cucial tool we use for proving our Lemma 2.1.

Let X and Y be Banach spaces and let $\{T_n\}$ and $\{R_n\}$ be sequences of continuous linear operators on X and Y, respectively. If S is a linear operator from X and Y such that $R_nS - ST_n$ is continuous for all n, then there is an integer N such that for $n \geq N$

$$(R_1 \ldots R_n\mathfrak{S}(S))^- = (R_1 \ldots R_n\mathfrak{S}(S))^-.$$

II. Continuity of Higher Derivations

A closed ideal J of a Banach algebra A is a separating ideal if for every sequence $\{a_n\}$ in A, there is an integer N such that for $n \geq N$

$$(Ja_n \ldots a_1)^- = (Ja_N \ldots a_1)^-.$$

By Stability Lemma we see that every derivation on a Banach algebra and every epimorphism from a Banach algebra onto a Banach algebra have separating spaces which are separating ideals [9]. Also we obtain the following lemma.

LEMMA 2.1. *Let $\{F_n\}$ be a higher derivation on a Banach algebra A onto a Banach algebra B and let for any k F_i be continuous for all $0 \leq i < k$. Then $\mathfrak{S}(F_k)$ is a separating ideal.*

PROOF: Since F_i is continuous for all $0 \leq i < k$, $\mathfrak{S}(F_k)$ is a closed ideal of B. Now let $\{b_n\}$ be any sequence in B and define R_n and T_n

by $R_n(y) = yb_n$ and $T_n(x) = xa_n$ for each n and x in A and y in B where $F_0(a_n) = b_n$. Then

$$\begin{aligned}(R_nF_k - F_kT_n)(x) &= F_k(x)b_n - F_k(xa_n) \\ &= F_k(x)b_n - \sum_{i=0}^{k} F_i(x)F_{k-i}(a_n) \\ &= -\sum_{i=0}^{k-1} F_i(x)F_{k-i}(a_n)\end{aligned}$$

Thus $R_nF_k - F_kT_n$ is continuous for each n. By Stability Lemma, $\mathfrak{S}(F_k)$ is a separating ideal.

THEOREM 2.2. *Let A be a Banach algebra and B a commutative semiprime Banach algebra with the radical R. If $\bigcap_{n\geq 1} R^n$ is contained in every closed prime ideal of B and $\{F_n\}$ is a higher derivation of any rank from A onto B such that if $F_0(x) \in R$ then $F_n(x) \in R$ for all n, then $\{F_n\}$ is continuous.*

PROOF: Since F_0 is onto homomorphism, F_0 is continuous by Theorem 12.1 in [**11**]. Now we assume that F_i is continuous for all $0 \leq i \leq k-1$. By Lemma 2.1, $\mathfrak{S}(F_k)$ is a separating ideal. Let $\pi : B \to B/R$ be a quotient map. Then $\{\pi \circ F_i\}$ is a higher derivation from A onto a semisimple Banach algebra B/R. By Theorem 7 in [**9**], $\{\pi \circ F_i\}$ is continuous, and so $\mathfrak{S}(\pi \circ F_k) = \overline{\pi\mathfrak{S}(F_k)} = (0)$. Thus $\mathfrak{S}(F_k) \subseteq R$. Let $\mathfrak{S}(F_k) = \mathfrak{S}$. Since $\mathfrak{S}$ is a separating ideal, there is m such that $\overline{x^m\mathfrak{S}} = \overline{x^n\mathfrak{S}}$, for all $n \geq m$ and $x \in \mathfrak{S} \subseteq R$. Then

$$\overline{x^m\mathfrak{S}} \supseteq \overline{\overline{x^m\mathfrak{S}}^2} \supseteq \overline{x^m\mathfrak{S}^2} = \overline{x^{2m}\mathfrak{S}\mathfrak{S}} \supseteq \overline{x^{m+m}x\mathfrak{S}} = \overline{x^m\mathfrak{S}}$$

and so

$$\overline{x^m\mathfrak{S}} = \overline{\overline{x^m\mathfrak{S}}^2}$$

By Mittag-Leffler Theorem [**1**, Theorem 3.3]

$$\overline{x^m\mathfrak{S}} = \overline{\bigcap_{n\geq 1} \overline{x^m\mathfrak{S}}^n} \subseteq \overline{\bigcap_{n\geq 1} R^n} \subseteq P$$

for every closed prime ideal P. Thus x^{m+1} belongs to every prime ideal P, and so $x \in P$, and note that every minimal prime ideal P where $\mathfrak{S} \not\supseteq P$ is closed [3, Lemma 2.3]. Thus

$$\begin{aligned} \mathfrak{S} &= \mathfrak{S} \cap \quad (\cap \text{ all closed prime ideal } P) \\ &\subseteq \mathfrak{S} \cap \quad (\cap \text{ all minimal prime ideal } P \text{ where } \mathfrak{S} \not\supseteq P) \\ &= \mathfrak{S} \cap N \end{aligned}$$

where N is the intersection of all prime ideal P. Since $N = (0)$, $\mathfrak{S} = (0)$. Thus F_k is continuous. By induction, we complete the proof.

Now we suppose that if $\{F_n\}$ is a higher derivation on a Banach algebra A, then $\{F_n\}$ is a higher derivation of any rank from A into A and F_0 is the identity map. The following lemma is similar to the Prime Ideal Theorem for the generalized intertwining operator which is contained in [1].

LEMMA 2.3. *Let $\{F_n\}$ be a higher derivation on a commutative semiprime Banach algebra A and F_i continuous for all $0 \le i \le k-1$. If F_i is discontinuous, then there is a discontinuous linear operator T_k on A and a minimal prime ideal P satisfying conditions*

(i) $T_k = zyF_k$ *for some* $z \in A \backslash P$ *and* $y \in A$
(ii) $\mathfrak{S}(T_k)$ *is a closed ideal*
(iii) *if* $a \in A$, *either* $\overline{a\mathfrak{S}(T_k)} = \mathfrak{S}(T_k)$ *or* $a\mathfrak{S}(T_k) = (0)$
(iv) $A(\mathfrak{S}(T_k)) = \{a \in A : a\mathfrak{S}(T_k) = (0)\}$ *is a minimal prime ideal,* $A(\mathfrak{S}(T_k)) \supseteq A(\mathfrak{S}(F_k))$ *and* $\mathfrak{S}(F_k) \not\subseteq P$
(v) $$\begin{aligned} &\{x \in A(\mathfrak{S}(T_k)) : F_i(x) \in A(\mathfrak{S}(T_k)), \text{ for all } 0 \le i \le k\} \\ &= \{x \in A(\mathfrak{S}(T_k)) : T_k(x) \in A(\mathfrak{S}(T_k))\} \\ &= A(\mathfrak{S}(T_k)). \end{aligned}$$

PROOF: By Lemma 2.1, $\mathfrak{S}(F_k)$ is a separating ideal and non-nilpotent. By Theorem 2.5 in [3], there is minimal prime ideal P such that $\mathfrak{S}(F_k) \not\subseteq P$, and P is closed. For all $z \in A \backslash P$, $\mathfrak{S}(F_k)z \not\subseteq P$ and so $\overline{\mathfrak{S}(F_k)z} \neq (0)$. Suppose that there is no element $z \in A \backslash P$ such that $\overline{az\mathfrak{S}(F_k)} = \overline{z\mathfrak{S}(F_k)}$ if $a \in A$ and $az \notin P$. Then there must be a sequence $\{z_n\}$ in $A \backslash P$ such that for all n

$$(0) \neq \overline{\mathfrak{S}(F_k)z_{n+1}z_n \ldots z_1} \subsetneq \overline{\mathfrak{S}(F_k)z_n \ldots z_1}$$

This is impossible since $\mathfrak{S}(F_k)$ is a separating ideal. Let $z \in A\backslash P$ such that if $a \in A$ and $az \notin P$ then $\overline{az\mathfrak{S}(F_k)} = \overline{z\mathfrak{S}(F_k)}$. Since P is prime, $\mathfrak{S}(F_k)zA \not\subseteq P$ and so $\mathfrak{S}(F_k)zA \neq (0)$. Let $\mathfrak{S}(F_k)zy_1 \neq (0)$. Suppose that there is no element y of A satisfying conditions (1) $\mathfrak{S}(F_k)zy \neq (0)$ and (2) if $a \in A$, $\overline{\mathfrak{S}(F_k)azy} = \overline{\mathfrak{S}(F_k)zy}$ or $\mathfrak{S}(F_k)azy = (0)$. Then there is $y_2 \in A$ such that $(0) \neq \overline{\mathfrak{S}(F_k)y_2zy_1} \subsetneq \overline{\mathfrak{S}(F_k)zy_1}$ and consequently there is a sequence $\{y_n\}$ in A such that for each n

$$(0) \neq \overline{\mathfrak{S}(F_k)zy_{n+1}\cdots y_1} \subsetneq \overline{\mathfrak{S}(F_k)zy_n\cdots y_1}$$

Since $\mathfrak{S}(F_k)$ is a separating ideal, it is impossible and so an element y with the required properties (1) and (2) must exist

(i) Let $T_k = zyF_k$. Then $T_k : A \to A$ is discontinuous because $\overline{\mathfrak{S}(F_k)zy} = \mathfrak{S}(zyF_k) = \mathfrak{S}(T_k) \neq (0)$

(ii) $\mathfrak{S}(T_k)$ is a closed ideal because F_i is continuous for all $0 \leq i \leq k-1$.

(iii) If $a \in A$, $\overline{a\mathfrak{S}(T_k)} = \overline{\mathfrak{S}(F_k)azy} = \overline{\mathfrak{S}(F_k)zy} = \mathfrak{S}(T_k)$ or $\overline{\mathfrak{S}(F_k)azy} = a\mathfrak{S}(T_k) = (0)$.

(iv) If $c_1 \notin A(\mathfrak{S}(T_k))$ and $c_2 \notin A(\mathfrak{S}(T_k))$ then $\overline{c_1c_2\mathfrak{S}(T_k)} \supseteq \overline{c_1c_2\mathfrak{S}(T_k)} = c_1\mathfrak{S}(T_k) \neq (0)$ and so $c_1c_2 \notin A(\mathfrak{S}(T_k))$. Thus $A(\mathfrak{S}(T_k))$ is a prime ideal. Since A is a semiprime Banach algebra $\mathfrak{S}(T_k) \not\subseteq A(\mathfrak{S}(T_k))$. If $\mathfrak{S}(T_k)$ is contained every prime ideal P, $\mathfrak{S}(T_k) \cap N = \mathfrak{S}(T_k) = (0)$ and it is impossible. Therefore there is a minimal prime ideal P such that $\mathfrak{S}(T_k) \not\subseteq P$, by Zorns Lemma. Since $\mathfrak{S}(T_k)A(\mathfrak{S}(T_k)) = (0) \subseteq P$, either $\mathfrak{S}(T_k) \subseteq P$ or $A(\mathfrak{S}(T_k)) \subseteq P$ and so $A(\mathfrak{S}(T_k)) \subseteq P$. Since $A(\mathfrak{S}(T_k))$ is a prime ideal such that $\mathfrak{S}(T_k) \not\subseteq A(\mathfrak{S}(T_k))$, $A(\mathfrak{S}(T_k)) = P$.

(v) Let $\{x \in A(\mathfrak{S}(T_k)) : F_i(x) \in A(\mathfrak{S}(T_k)) \text{ for all } 0 \leq i \leq k\} = E$. Then $E \supseteq \{x \in A(\mathfrak{S}(T_k)) : T_k(x) \in A(\mathfrak{S}(T_k))\} \supseteq A(\mathfrak{S}(T_k))$ and $\mathfrak{S}(T_k) \not\subseteq E$. If $x \in E$ and $y \in A$, then

$$F_n(xy) = \sum_{i=0}^{n} F_i(x)F_{n-i}(y) \in A(\mathfrak{S}(T_k))$$

and so $xy \in E$. Suppose that $xy \in E$ and $y \notin E$ for any x, y, in A. Then we can prove easily that E is a prime ideal where $\mathfrak{S}(T_k) \not\subseteq E$. Therefore $E = \{x \in A(\mathfrak{S}(T_k)) : T_k(x) \in A\mathfrak{S}(T_k))\} = A(\mathfrak{S}(T_k))$.

THEOREM 2.4. *The following conditions are equivalent.*

(i) *Every higher derivation on a commutative Banach algebra which is an integral domain is continuous.*

(ii) *Every higher derivation on a commutative semiprime Banach algebra is continuous.*

PROOF: Obviously (ii) implies (i). So assume (i). Suppose (ii) is false. Then there is a discontinuous higher derivation $\{F_n\}$ on some semiprime Banach algebra A. We may assume that F_i is continuous for all $0 \leq i \leq k-1$ and F_k is discontinuous. By Lemma 2.3, we can define a linear operator

$$\overline{F}_i : A/A(\mathfrak{S}(T_k)) \longrightarrow A/A(\mathfrak{S}(T_k))$$

by $\overline{F}_i(a + A(\mathfrak{S}(T_k))) = F_i(a) + A(\mathfrak{S}(T_k))$ for all $0 \leq i \leq k$ and a in A. Then $\{\overline{F}_i\}$ is a higher derivation on an integral domain and so it is continuous. In particular, $\overline{F}_k$ is continuous. By Lemma 1.4 in [**12**], $\mathfrak{S}(T_k) \subseteq \mathfrak{S}(F_k) \subseteq A(\mathfrak{S}(T_k))$. Thus $\mathfrak{S}(T_k)$ is nilpotent. It is impossible. Therefore we complete the proof.

THEOREM 2.5. *Let A be a commutative semiprime Banach algebra in which every prime ideal is closed. Then every higher derivation on A is continuous.*

PROOF: Suppose the theorem is false. Then there is a discontinuous higher derivation $\{F_n\}$ on A. We may assume that F_k is discontinuous and F_i continuous for all $0 \leq i \leq k-1$. By Lemma 2.3, we can define a discontinuous higher derivation $\{\overline{F}_i\}$ on $A/A(\mathfrak{S}(T_k))$. Also $A/A(\mathfrak{S}(T_k))$ is an integral domain and every prime ideal in $A/A(\mathfrak{S}(T_k))$ is closed. Now we identify A with $A/A(\mathfrak{S}(T_k))$ and identify F_i with $\overline{F}_i$ for each i. Note that T_k is a discontinuous linear operator, $A(\mathfrak{S}(T_k))$ is a minimal prime ideal and for any a in A either $\overline{a\mathfrak{S}(T_k)} = \mathfrak{S}(T_k)$ or $\overline{a\mathfrak{S}(T_k)} = (0)$. Hence, if I is a closed nonzero ideal and $0 \neq x \in I$, then $\overline{x\mathfrak{S}(T_k)} = \mathfrak{S}(T_k) \subseteq I$. Thus $\mathfrak{S}(T_k)$ is contained in every nonzero prime ideal. Also $\mathfrak{S}(T_k) \subseteq R$.

If I is a nonzero ideal and if $\mathfrak{S}(x) = \{x, x^2, \dots\}$ where $x \in \mathfrak{S}(T_k)$, then if $I \cap \mathfrak{S}(x) = \emptyset$, we can find a prime ideal $P \supseteq I$ with $\mathfrak{S}(x) \cap P = \emptyset$. But P is a closed ideal and this contradicts the previous observation. Hence $\mathfrak{S}(x) \cap I \neq \emptyset$ for every non-zero ideal I of A.

Now since $\overline{x\mathfrak{S}(T_k)} = \mathfrak{S}(T_k)$ for every $x \neq 0$, the argument of Theorem 8.1 in [4] applies and produce the existence of two element b, $c \in \mathfrak{S}(T_k)$ such that $b^n \notin cA$ and $c^n \notin bA$, $n = 1, 2, \ldots$. This means $\mathfrak{S}(b) \cap cA = \emptyset$ and $\mathfrak{S}(c) \cap bA = \emptyset$. This contradicts the previous paragraph and proves that $\{\overline{F}_n\}$ is continuous. Thus for the original higher derivation $\{F_n\}$, $\mathfrak{S}(T_k) \subseteq A(\mathfrak{S}(T_k))$, and so $\mathfrak{S}(T_k)^2 = (0)$. But A is semiprime, so $\mathfrak{S}(T_k) = (0)$. This is a contradiction.

REMARK: The proof of Theorem 2.5 is similar to that of Theorem 3.2 in [5]. In Theorem 2.5, we use the induction and it is the case of higher derivation. Banach algebras satisfying the hypothesis of the theorem occur in the work of Sandy Grabiner [7].

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