JOURNAL OF THE
CHUNGCHEONG MATHEMATICAL SOCIETY
Volume 2, June 1989

A Singular Nonlinear Boundary Value Problem

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ABSTRACT. Certain type of singular two point boundary value problem is studied. This contains a wider class of differential equations than [5]. An example is provided for comparison with earlier results.

1. Introduction

In this paper we show the existence of the solution of a singular boundary value problem of the form

$$y'' + f(t,y) = 0 \tag{1}$$

$$y(0) = y(1) = 0, \tag{2}$$

where $f(t,y) = \sum_{i=1}^{n} \phi_i(t) g_i(y)$, ϕ_i positive, continuous for $0 < t < 1$, $g_i(y)$ positive and decreasing. This problem is singular because f may have singularities at both end points $t = 0$, $t = 1$ and $y = 0$. Equation (1) with $f(t,y) = \phi(t)y^{-\lambda}$, $\lambda < 0$, $\phi(t) > 0$ continuous, which is known as generalized Emden-Fowler equation, has been studied for a longtime, see [3], for example. The case $\lambda > 0$ arises in fluid dynamics, and is done in [5]. Similar problems were considered in [2]. Our result generalizes Theorem 1 of [5] and even though Theorem 2 of [2] seems to be somewhat more general than our result, it is sometimes very difficult to check the conditions imposed on that theorem. Later we present an example for which this is so.

Received by the editors on 20 April 1989.
1980 *Mathematics subject classifications*: Primary 34.

2. Existence

We need a lemma whose proof can be easily obtained by slightly modifying that in [4].

LEMMA. *Let $\phi_i(t)$ and $\psi_i(t)$ are nonnegative and continuous on $[a,b]$ with $\phi_i(t) \le \psi_i(t)$. Suppose $y(t)$ and $z(t)$ are positive solutions of $y'' + \sum \phi_i(t)g_i(y) = 0$ and $z'' + \sum \psi_i(t)g_i(y) = 0$ on $[a,b]$ resp. If $y(a) \ge z(a)$ and $y'(a) > z'(a)$ (resp. $y(b) \ge z(b)$ and $y'(b) < z'(b)$) then $y(t) > z(t)$ and $y'(t) > z'(t)$ on $[a,b]$ (resp. $y(t) > z(t)$ and $y'(t) < z'(t)$ on $[a,b)$).*

THEOREM. *The equation (1), (2) has a unique solution if and only if*

$$\int_0^1 t(1-t)\phi_i(t)\,dt < \infty, \qquad i = 1, \dots, n \tag{3}$$

PROOF: Suppose y is a solution. Integrating (1), we obtain

$$y'(t) - y'\left(\tfrac{1}{2}\right) + \sum \int_{\frac{1}{2}}^{t} \phi_i(s)g_i(y)\,ds = 0.$$

Integrating again and using Fubini's Theorem, we obtain

$$y(t) - y\left(\tfrac{1}{2}\right) - y'\left(\tfrac{1}{2}\right)\left(t - \tfrac{1}{2}\right) + \sum \int_{\frac{1}{2}}^{t} (t-s)\phi_i(s)g_i(y(s))\,ds = 0 \tag{4}$$

we let $t \to 1^-$. Then by the Monotone Convergence Theorem the integral approaches

$$\sum \int_{1/2}^{1} (1-s)\phi_i(s)g_i(y(s))\,ds = y(1) - y\left(\tfrac{1}{2}\right) - \tfrac{1}{2}y'\left(\tfrac{1}{2}\right) < \infty$$

Thus $\int_{1/2}^{1} (1-s)\phi_i(s)\,ds < \infty$. Similarly, $\int_0^{1/2} s\phi_i(s)\,ds < \infty$. Conversely, suppose (3) holds. Given $a > 0$, we claim that there exists a solution $v_a(t)$ of (1) on $\left[\frac{1}{2}, 1\right)$ such that $v_a\left(\frac{1}{2}\right) = a$ and $\lim_{t\to 1} v_a(t) = 0$. If such $v_a(t)$ exists, it is unique by Lemma. We show that $v_a(t)$ is given as a solution of the initial value problem whose existence is well known:

$$y_\alpha'' + \sum \phi_i g_i(y_\alpha) = 0$$
$$y_\alpha\left(\tfrac{1}{2}\right) = a, \quad y_\alpha'\left(\tfrac{1}{2}\right) = \alpha \quad \text{for some } \alpha.$$

We choose α so that $\lim_{t\to 1} y_\alpha(t)$ attains infimum. This method resembles well known shooting method. Since $y'_{-2a}(\frac{1}{2}) = -2a$ and $y''_{-2a} < 0$, we have, by Lemma $y_{-2a}(t) < -2a(t-1)$, $\frac{1}{2} \le t < 1$. Hence $y_{-2a}(t_0) = 0$ for some $\frac{1}{2} < t_0 < 1$. On the other hand, suppose for every α there is $\frac{1}{2} < t_\alpha < 1$ such that $y_\alpha(t_\alpha) = a$ and $y_\alpha(t) > a$, $\frac{1}{2} < t < t_\alpha$. Integrating (1) twice from $\frac{1}{2}$ to t_α, we get

$$0 = y_\alpha(t_\alpha) - y_\alpha\left(\tfrac{1}{2}\right) = \alpha\left(t_\alpha - \tfrac{1}{2}\right) - \int_{1/2}^{t_\alpha} (t_\alpha - s)\sum \phi_i(s) g_i(y_\alpha(s))\, ds.$$

Hence

$$\text{(5)} \qquad \begin{aligned} \alpha\left(t_\alpha - \tfrac{1}{2}\right) &\le \max_i g_i(a) \int_{1/2}^{t_\alpha} (t_\alpha - s)\sum \phi_i(s)\, ds \\ &\le \max_i g_i(a) \sum_i \int_{1/2}^{1} (1-s)\phi_i(s)\, ds \end{aligned}$$

By Lemma, t_α increases as α increases, hence (4) tend to ∞, a contradiction. Hence for some α, $y_\alpha(t) > 0$ for $\frac{1}{2} \le t < 1$. Let $F : \mathbf{R} \to C^2\left[\frac{1}{2}, 1\right)$ be the operator $\alpha \mapsto y_\alpha$. We claim F is continuous. Let $\alpha_n > \alpha$, $\alpha_n \to \alpha$. By Lemma $F(\alpha_n)(t) > F(\alpha_{n+1})(t)$, letting $n \to \infty$ with y_{α_n} in place of y_α in (4), we get by Monotone Convergence Theorem

$$\text{(6)} \qquad y_\alpha(t) = a + \alpha\left(t - \tfrac{1}{2}\right) - \int_{1/2}^{t} (t-s)\sum \phi_i(s) g_i(y_\alpha(s))\, ds.$$

It follows that y_α is $C^2\left[\frac{1}{2}, 1\right)$, hence F is left continuous. Similarly, F is right continuous. If we let $\alpha_0 = \inf\left\{\alpha : y_\alpha(t) > 0, \ \ \frac{1}{2} \le t < 1\right\}$, then by Lemma $\alpha_0 \ge \beta$, for all β such that $y_\beta(t_0) < 0$ for some

$\frac{1}{2} < t_0 < 1$, and there exists a sequence $\alpha_n \downarrow \alpha_0$ and $\beta_n \uparrow \alpha_0$ such that $\lim_{n\to\infty} y_{\alpha_n}(t) = \lim_{n\to\infty} y_{\beta_n}(t)$ exist. Then clearly, $y_{\alpha_0}(t) > 0$, $\frac{1}{2} \le t < 1$ and $\lim_{t\to 1^-} y_{\alpha_0}(t) = 0$. We call this $v_\alpha(t)$. By a similar argument, one can show there is a unique solution $u_a(t)$ of (1) on $\left(0, \frac{1}{2}\right]$ such that $u_a\left(\frac{1}{2}\right) = a$ and $\lim_{t\to 0+} u_a(t) = 0$. By Lemma, $v'_a\left(\frac{1}{2}\right)$ is a decreasing function of a, for $a > 0$. We show $v'_a\left(\frac{1}{2}\right)$ is a continuous function of a. Let $a_0 > 0$ and $m_1 = \lim_{a\to a_0^-} v'_a\left(\frac{1}{2}\right)$, $m_2 = \lim_{a\to a_0^+} v'_a\left(\frac{1}{2}\right)$ so that $m_1 \le m_2$. By the continuity of solutions with respect to the initial conditions, $v_a(t)$ converges to the solution $v_{a_0}(t, m_1)$ of (1) with the initial conditions

$$y\left(\tfrac{1}{2}\right) = a_0, \qquad y'\left(\tfrac{1}{2}\right) = m_1,$$

as $a \uparrow a_0$. If $v'_{a_0}\left(\frac{1}{2}\right) < m_1$, then by Lemma $v_{a_0}(t) < v_{a_0}(t, m_1)$, which is a contradiction to the fact that $v_a(t) \to v_{a_0}(t, m_1)$ as $a \uparrow a_0$. Thus $v'_{a_0}\left(\frac{1}{2}\right) \ge m_1$. Since $v_a''(t) < 0$, $v_a(t) \ge 2a(1-t)$ on $\left[\frac{1}{2}, 1\right)$ for all $a > a_0$, we have $v_{a_0}(t, m_2) \ge 2a(1-t)$. If $v'_{a_0}(\frac{1}{2}) > m_2$ we have $v_{a_0}(1) > v_{a_0}(1, m_2) \ge 0$, a contradiction. Hence $m_2 \ge v'_{a_0}\left(\frac{1}{2}\right)$. Thus $m_1 = m_2 = v'_{a_0}\left(\frac{1}{2}\right)$. This shows $v'_a\left(\frac{1}{2}\right)$ is a continuous function of a.

Next step is to choose a judiciously so that $v'_a\left(\frac{1}{2}\right) = u'_a\left(\frac{1}{2}\right)$ and hence u, v piece together to be a solution. Integrating (1) twice and letting $t \to 1$ we get

$$\int_{1/2}^{1} (1-s) \sum \phi_i(s) g_i(v_a(s))\, ds = a + \tfrac{1}{2} v'_a\left(\tfrac{1}{2}\right).$$

If $v'_a\left(\frac{1}{2}\right) \le 0$ then $v_a(s) \le a$ for $\frac{1}{2} \le s < 1$ and hence

$$\min_i g_i(a) \int_{1/2}^{1} (1-s) \sum \phi_i(s)\, ds \le a$$

or

$$\int_{1/2}^{1} (1-s) \sum \phi_i(s)\, ds \le \frac{a}{\min_i g_i(a)}.$$

But since the left hand side is independent of a, this leads to a contradiction. Hence $v'_a\left(\frac{1}{2}\right)$ must be positive for small a. Similarly,

$u'_a\left(\frac{1}{2}\right) < 0$ for all sufficiently large a. Since $W(a) \equiv v'_a\left(\frac{1}{2}\right) - u'_a\left(\frac{1}{2}\right)$ is a continuous function of a, we can use the intermediate value theorem to conclude that there exists an a such that $v'_a\left(\frac{1}{2}\right) = u'_a\left(\frac{1}{2}\right)$. Since $v_a''\left(\frac{1}{2}\right) = u''_a\left(\frac{1}{2}\right) = -f\left(\frac{1}{2}, a\right)$, piecing u, v to together, we obtain the solution of (1) and (2).

3. Comparison with Other Results

Taliafero gave the following simple version in [**5**].

THEOREM A. *The boundary value problem*

$$y'' + \phi(t)y^{-\lambda} = 0$$
$$y(0) = y(1) = 0$$

where $\lambda > 0$, $\phi(t) > 0$ *and continuous has a solution if and only if*

$$\int_0^1 t(1-t)\phi(t)\,dt < \infty.$$

Although our proof uses the method of Taliafero, our result is obviously more general than this. Next we present the result of Bobisud et. al [**5**] which is a generalization of Theorem A.

THEOREM B. *Let* (P) *denote the boundary value problem*

$$y'' + f(t,y) = 0, \qquad 0 < t < 1,$$
$$y(0) = y(1) = 0$$

Suppose that

(a) $f(t,y)$ *is continuous and positive on* $(0,1) \times (0,\infty)$*;*
(b) $f(t,y)$ *is strictly decreasing in* y*;*
(c) *for some constant* k,

$$\lambda f(t,\lambda y) \le k f(t,y)$$

for $0 < \lambda \le 1$, $0 < t < 1$, *and* $y > 0$*;*
(d) *there exists a nonnegative* $\alpha(t)$ *satisfying*
   (i) $\alpha''(t) + f(t,\alpha(t)) \ge 0$ *on* $(0,1)$, $\alpha(0) = \alpha(1) = 0$
   (ii) $\int_0^1 f(t,\alpha(t))\,dt < \infty$,
   (iii) $f(t,y)/f(t,\alpha(t))$ *is continuous on* $[0,1] \times (0,\infty)$.

Then (P) possesses a solution.

Their proof is quite long using topological transversality theorem, and not elementary. Even though Theorem B can be applied to such problem as

$$f(t,y) = t^{-1/2}y^{-1/4} + t^{1/2}(1-t)^{1/2}y^{-1}$$

with $\alpha(t) = ct(1-t)$, c small, it is not applicable to the following case

$$f(t,y) = t^{-1/2}(1-t)^{-1/2}y^{-1} + t(1-t)y^{-1/2}.$$

Because if one chooses $\alpha(t) = ct(1-t)$, $f(t,\alpha(t)) \geq t^{-\frac{3}{2}}(1-t)^{-\frac{3}{2}}$ is not intergrable. But our result can be applied without any trouble.

References

1. Bobisud L.E., O'Reagan D. and Royalty W.D., *Singular boundary value problems*, Appl. Anal. **23** (1986), 233–243.
2. ——————, *Solvability of some nonlinear boundary value problems*, Nonlinear Anal. **12** (1988), 855–869.
3. Wong J.S.W., *On the generalized Emden-Fowler equation*, SIAM Rev. **17** (1975), 339–360.
4. Taliafero S., *On the positive solutions of* $y'' + \phi(t)y^{-\lambda} = 0$, Nonlinear Anal. **2** (1978), 437–446.
5. ——————, *A nonlinear singular boundary value problem*, Nonlinear Anal. **3** (1979), 877–904.

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