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Ersatz Chern Polynomials*

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ABSTRACT. Some kinematic formulas of the Ersatz Chern polynomial and the generalized volume function are derived.

1. Introduction

In [4] Gray defined the Ersatz Chern polynomial $k(P,t)$ for all compact Riemannian manifolds P. This polynomial reflects many properties of Chern forms of a Kähler manifold. The polynomial $k(P,t)$ arises natually from the study Weyl's tube formula. The following formulas ([4]) express the remarkable properties of the Ersatz Chern polynomial.

Let P and Q be Riemannian manifolds for which the Ersatz Chern polynomial is defined. Then

$$(1) \qquad k(P \times Q, t) = k(P,t)k(Q,t),$$

$$(2) \qquad k(\widetilde{P}, t) = sk(P,t).$$

Here $P \times Q$ is the Reimannian product of P and Q, and $\widetilde{P}$ is a s-fold covering $\widetilde{P} \to P$.

The Ersatz Chern polynomial also has a simple relation with the generalized volume functions ([4])

$$(3) \qquad k(P,t) = e^{-\pi r^2} \sum_{n-p=\text{even}} V_P^{\mathbf{R}^n}(r,t).$$

To explain $k(P,t)$ and $V_P^{\mathbf{R}^n}(r,t)$ let us look at Weyl's tube formula ([8]) for the volume of the tube of radius r about a compact

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p-dimensional submanifold P of $\mathbf{R}^n$ with the curvature tensor R^P (briefly $P \subset \mathbf{R}^n$)

$$V_P^{\mathbf{R}^n}(r) = \sum_{c=0}^{[\frac{p}{2}]} \frac{k_{2c}(R^P)(\pi r^2)^{\frac{1}{2}(n-p)+c}}{(2\pi)^c(\frac{1}{2}(n-p)+c)!}. \tag{4}$$

Then the *Ersatz Chern polynomial* $k(R^P, t)$ (briefly $k(P, t)$) is defined by

$$k(R^P, t) = \sum_{c=0}^{[\frac{p}{2}]} k_{2c}(R^P)t^c \tag{5}$$

and the *generalized volume function* is defined by

$$V_P^{\mathbf{R}^n}(r, t) = \sum_{c=0}^{[\frac{p}{2}]} \frac{k_{2c}(R^P)t^c(\pi r^2)^{\frac{1}{2}(n-p)+c}}{(2\pi)^c(\frac{1}{2}(n-p)+c)!}. \tag{6}$$

For the definition of integral invariant $k_{2c}(R^P)$ (briefly $k_{2c}(P)$) see (12) in § 2.

It is important to observe that if P is not given as a submanifold of $\mathbf{R}^n$ then (4) can be regarded as a *definition*, and (3) should be read with this interpretation.

In this article we study the Ersatz Chern polynomial and the generalized volume function from the integro-geometric point of view. We shall prove the following.

THEOREM 1. *Let $P \subset \mathbf{R}^n$ and $Q \subset \mathbf{R}^n$ be compact manifolds. Let dg be the standard kinematic density on the group of proper motions of $\mathbf{R}^n$. If $0 \leq 2c \leq p+q-n$, then*

$$\int k(P \cap gQ, t)\, dg = \sum_{c=0}^{[\frac{p+q-n}{2}]} \sum_{i=0}^{c} d_{c,i} k_{2i}(P) k_{2c-2i}(Q) t^c. \tag{7}$$

with constants $d_{c,i}$ depending on p, q, n, c and i (see the formula (7) in § 3). We also have

$$\sum_{p+q-n-m=\text{even}} \int V^{\mathbf{R}^m}_{p\cap gQ}(r,t)\,dg = e^{\pi r^2} \sum_{c=0}^{[\frac{p+q-n}{2}]} \sum_{i=0}^{c} d_{c,i} k_{2i}(P) k_{2c-2i}(Q) t^c. \tag{8}$$

The second purpose of this article is to study the polynomial $k(R^P - K, t)$ for $P \subset \mathbf{E}^n(K)$, where $\mathbf{E}^n(K)$ is n-dimensional non-Euclidean space of constant curvature K (with the curvature tensor $R^{\mathbf{E}^n(K)}$, briefly K). In this case $k(R^P - K, t)$, which is still called the *Ersatz Chern polynomial*, is defined by

$$k(R^P - K, t) = \sum_{c=0}^{[\frac{p}{2}]} k_{2c}(R^P - K) t^{2c}. \tag{9}$$

Then we have the following.

THEOREM 2. *For $P \subset E^n(K_1)$ and $Q \subset E^n(K_2)$*

$$k(R^{P\times Q} - K_1 \times K_2, t) = k(R^P - K_1, t) k(R^Q - K_2, t). \tag{10}$$

THEOREM 3. *Let dg be the standard kinematic density on the group of proper motions of $\mathbf{E}^n(K)$. Let $P \subset \mathbf{E}^n(K)$ and $Q \subset E^n(K)$. If $0 \le 2c \le p+q-n$, then*

$$\int k(R^{P\cap gQ} - K, t)\,dg = \sum_{c=0}^{[\frac{p+q-n}{2}]} \sum_{i=0}^{c} d_{c,i} k_{2i}(R^P - K) k_{2c-2i}(R^Q - K) t^c. \tag{11}$$

REMARK: The kinematic density dg in (7) is normalized so that the total measure of the group of proper motions of $\mathbf{R}^n$ is equal to $O_n O_{n-1} \cdots O_2$, where $O_n = \dfrac{2\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})}$ is the volume of the unit sphere $S^{n-1}(1)$ in $\mathbf{R}^n$. Similarly dg in (11) is normalized so that $\int dg = O_n O_{n-1} \cdots O_2$ (see [1] for details).

2. Preliminaries

Let P be a compact p-dimensional Reimannian manifold with the Riemannian curvature tensor R^P. To explain the integral invariants $k_{2c}(R^P)$ (briefly $k_{2c}(P)$), $0 \leq 2c \leq p$, it will be convenient to use the notations of [**2**, **3**].

Let R be a tensor field on P of the same type as the curvature tensor field on P and having the same symmetries. R is called a *curvaturelike* tensor field on P. It is possible to define the c-th power of R. The definition can be given inductively via $R^0 = 1$ and

$$R^c(x_1 \wedge \cdots \wedge x_{2c})(y_1 \wedge \cdots \wedge y_{2c}) = \sum_{i,j,k,l=1}^{2c} (-1)^{i+j+k+l} R_{x_i x_j x_k x_l} R^{c-1}(x_1 \wedge \cdots \wedge \hat{x}_i \wedge \cdots \wedge \hat{x}_j \wedge \cdots \wedge x_{2c})(y_1 \wedge \cdots \wedge \hat{y}_k \wedge \cdots \wedge \hat{y}_l \wedge \cdots \wedge y_{2c}),$$

where $x_1, \ldots, y_{2c}$ are tangent vectors to P. Then the complete contraction of R^c is

$$C^{2c}(R^c) = \sum_{a_1,\ldots,a_{2c}=1}^{p} R^c(e_{a_1} \wedge \cdots \wedge e_{a_{2c}})(e_{a_1} \wedge \cdots \wedge e_{a_{2c}}),$$

where $\{e_1, \ldots, e_p\}$ is any orthonormal frame on P.

We put

$$k_{2c}(R) = \frac{1}{c!\,(2c)!} \int_P C^{2c}(R^c)\, dP, \tag{12}$$

where dP is the volume element of P.

Next we recall Chern's kinematic formula in Euclidean space ([**1**]). Let P and Q be two compact embedded submanifolds of $\mathbf{R}^n$ of respective dimensions p and q with $p + q - n \geq 0$. Let dg be the standard kinematic density on the group of proper motions of $\mathbf{R}^n$. Chern considered the integral invariants $\mu_e(R^P)$ (briefly $\mu_e(P)$) which are related by

$$\mu_e(P) = \frac{2^{\frac{e}{2}}(p-e)!\,\left(\frac{e}{2}\right)!}{p!} k_e(P). \tag{13}$$

If $0 \leq e$ even $\leq p+q-n$, then

$$\int \mu_e(P \cap gQ)\, dg = \sum_{0 \leq i \text{ even } \leq e} c_i \mu_i(P) \mu_{e-i}(Q), \tag{14}$$

where constants c_i depending only on p, q, n and e are given by ([7])

$$c_i = \frac{O_{n+1} \cdots O_2 \left(\dfrac{O_{p+q-n+1} O_{p+q-n+2} (\frac{e}{2})!}{O_{p+q-n-e+2}} \right)}{\left(\dfrac{O_{p+1} O_{p+2} (\frac{i}{2})!}{O_{p-i+2}} \right) \left(\dfrac{O_{q+1} O_{q+2} \left(\frac{e-i}{2}\right)!}{O_{q-e+i+2}} \right)}. \tag{15}$$

There is also a non-Euclidean version of Chern's kinematic formula ([6]). Let P and Q be two compact embedded submanifolds of $\mathbf{E}^n(K)$ and dg the standard kinematic density on the group of proper motions of $\mathbf{E}^n(K)$. If $0 \leq e$ even $\leq p+q-n$, then

$$\int \mu_e(R^{P \cap gQ} - K)\, dg = \sum_{0 \leq i \text{ even } \leq e} c_i \mu_i(R^P - K) \mu_{e-i}(R^Q - K), \tag{16}$$

where constants c_i are given by (14). Here $\mu_e(R^P - K)$ is defined by $\mu_e(R^P)$ except that R^P is replaced by $R^P - K$. Note that $R^P - K$ is a curvaturelike tensor field on $P \subset \mathbf{E}^n(K)$.

3. Proof of Theorems

Proof of Theorem 1. Let $P \subset \mathbf{R}^n$ and $Q \subset \mathbf{R}^n$. In order to derive

(7) we combine (5) and (14) using (13) and (15). Then we have

$$\begin{aligned}
&\int k(P \cap gQ, t)\, dg \\
&= \sum_{c=0}^{[\frac{p+q-n}{2}]} t^c \int k_{2c}(P \cap gQ)\, dg \\
&= \sum_{c=0}^{[\frac{p+q-n}{2}]} \frac{t^c (p+q-n)!}{2^c (p+q-n-2c)!\, c!} \int \mu_{2c}(P \cap gQ)\, dg \\
&= \sum_{c=0}^{[\frac{p+q-n}{2}]} \frac{t^c (p+q-n)!}{2^c (p+q-n-2c)!\, c!} \sum_{i=0}^{c} c_i \mu_{2i}(P) \mu_{2c-2i}(Q) \\
&= \sum_{c=0}^{[\frac{p+q-n}{2}]} \sum_{i=0}^{c} d_{c,i} k_{2i}(P) k_{2c-2i}(Q) t^c,
\end{aligned}$$

where

(17)

$$d_{c,i} = \frac{O_{n+1} \cdots O_2 O_{p+q-n+1} O_{p+q-n+2} O_{p-2i+2} O_{q-2c+2i+2} (p+q-n)!\, (p-2i)!\, (q-2c+2i)!}{O_{p+q-n-2c+2} O_{p+1} O_{p+2} O_{q+1} O_{q+2} (p+q-n-2c)!\, p!\, q!}.$$

Next we combine (3) and (7) to show (8). Then we have

$$\begin{aligned}
&\sum_{p+q-n-m=\mathrm{even}} \int V^{\mathbf{R}^m}_{P \cap gQ}(r, t)\, dg \\
&= e^{\pi r^2} \int k(P \cap gQ, t)\, dg \\
&= e^{\pi r^2} \sum_{c=0}^{[\frac{p+q-n}{2}]} \sum_{i=0}^{c} d_{c,i} k_{2i}(P) k_{2c-2i}(Q) T^C.
\end{aligned}$$

Proof of Theorems 2 and 3. The formula (10) comes from

$$k_{2c}(R^{P \times Q} - K_1 \times K_2) = \sum_{i=0}^{c} k_{2i}(R^P - K_1) k_{2c-2i}(R^Q - K_2), \tag{18}$$

of which proof can be found, for example, in [**5**].

Finally the formula (11) can be obtained by combining (10) and (16). The derivation is similar to that of (7).

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