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Application of Volterra Series to Modeling an Elastomer Force-Displacement Relation

(고무의 힘-변위 관계를 나타내는 모델링에의 볼테라 급수의 응용)

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要 約

비선형 시스템의 입출력 관계는 볼테라 급수로 나타낼 수 있으며 그것은 볼테라 커널에 의해 규명될 수 있다. 본 연구는 소수의 볼테라 급수항을 사용하여 계단 입력에 의한 고무의 힘-변위 관계를 모델링하는데 관한 것이다. 무한의 기울기를 가진 계단 입력을 가하는 것은 불가능하므로 다른 방법으로 일정한 압축 속도와 일정한 압축의 혼합형 입력을 가하여 얻어진 실험 결과에서 일정한 압축시의 계단 입력에 대한 결과를 예측한다. 비스코일레스틱 재료를 나타내는 현저한 특징중의 하나인 힘-변위 관계를 모델링하기 위해 2차와 3차 볼테라 급수 모델을 사용한다. 3차 볼테라 급수 모델이 2차 볼테라 급수 모델의 결과에 비해 더 좋은 결과가 얻어지고 있다.

Abstract

The input-output relations for nonlinear systems can be explicitly represented by the Volterra series and they can be characterized by the Volterra kernels. This study is concerned with modeling an elastomer force-displacement relation due to step inputs by utilizing the truncated Volterra series. Since it is practically impossible to apply step inputs that have infinite slope at zero time, the loads due to constant penetration (displacement) rate followed by constant penetration inputs are measured as an alternative approach and estimated for step inputs and then utilized for the truncated Volterra series models. One second order and one third order truncated Volterra series models have been employed to model the force-displacement relation which is one of the prominent properties to characterize the viscoelastic material. The third order truncated Volterra series model has better results, compared with those of the second order truncated Volterra series model.

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I. Introduction

Since Vito Volterra[1] introduced an infinite functional series around 1910, which is now referred to as the Volterra series, to represent functionals which are analytic, the Volterra series has been applied in the wide range of nonlinear/bilinear systems such as, communications[2], circuits[3,4,5,6], viscoelastic material[7,8], identification[9], and mechanical systems[10]. The input-output relations for nonlinear analytic systems can be explicitly represented by the Volterra series and they can be characterized by the Volterra kernels. If the systems are not strongly nonlinear, they can be approximated by the truncated Volterra series.

Elastomers are viscoelastic polymers capable of undergoing reversible deformations. One of the most important characteristics of elastomers is the dependence of their mechanical properties on time. If an elastomer sample is constrained at constant deformation (strain), then the stress required to maintain that strain decreases with time. This phenomenon is known as stress relaxation.

So far several investigators[11,12,13] have made attempts to model viscoelastic constitutive relations by utilizing the Volterra series representations for step inputs. However, they have not applied pure step inputs to the material, because it is practically impossible to apply pure step inputs that have infinite slope at zero time. If the short time responses are of particularly interest, then we may not neglect the transient responses due to inaccurate step inputs. As an alternative approach we use combined inputs with constant penetration rate followed by constant penetration inputs and then estimate the responses due to pure step inputs from the obtained results due to combined inputs.

This study is concerned with modeling an elastomer force-displacement relation due to step inputs by utilizing the truncated Volterra series. In chapter 2, the Volterra series and multiple integral representations are compared and some erroneous treatments of kernels in the material science field are also discussed. In chapter 3, one elastomer constitutive relation is represented by the one-dimensional Volterra series model. In chapter 4, the force-displacement relations are modeled by one second order and one third order truncated Volterra series models, and they are analyzed for step inputs.

II. Volterra Series Representations Versus Multiple Integral Representations

The input-output relations for nonlinear time-invariant analytic systems can be explicitly represented by the Volterra series with the following form:

$$y(t)=\sum_{n=1}^{\infty}H_n[x(t)], \tag{1}$$

where

$$H_n[x(t)]=\int_0^t\cdots\int_0^t h_n(t_1,t_2,\cdots,t_n)\,x(t-t_1)\cdots x(t-t_n)\,dt_1\cdots dt_n.$$

H_n is called the n-th order Volterra operator and $H_n[x(t)]$ is expressed as an n-dimensional generalized convolution integral containing the n-th order kernel multiplied by an n-th order product of the forcing functions. The n-th order Volterra kernel is denoted by h_n $(t_1, t_2, ..., t_n)$ and is also called the generalized impulse response. In particular, the linear systems can be characterized only by the first order Volterra kernels. The above input-output relation is called the Volterra series representation in the electrical engineering field.

In the material science field, one constitutive relation can be described by

$$\sigma(t)=\sum_{n=1}^{\infty}G_n(\dot{\varepsilon}(t)], \tag{2}$$

where

$$G_n[\dot{\varepsilon}(t)]=\int_{-\infty}^t\cdots\int_{-\infty}^t g_n(t-t_1,\cdots,t_n)\,\dot{\varepsilon}(t_1)\cdots\dot{\varepsilon}(t_n)\,dt_1\cdots dt_n.$$

The stress, strain rate, and stress relaxation function are denoted by $\sigma(t)$, $\dot{\epsilon}(t)$, and g_n $(t-t_1, ..., t-t_n)$, respectively. This input-output relation is called the multiple integral representation.

Both representations (1) and (2) are equivalent and have the same historical origin. However, they have been developed differently and independently in the above mentioned electrical engineering and material science fields. There have been several erroneous treatments of kernels in the multiple integral representations. Several investigators made attempts to find the high order kernels after assuming particularly convenient forms for the estimation of the kernel functions. However, most assumptions appear to be inappropriately restrictive and they give rise to erroneous results. For example, Gottenberg et al[8] assumed special forms of the kernel functions in which the argument of these functions are taken in additive form, i.e.

$$\Psi_k(t-t_1, t-t_2, \cdots, t-t_k) = f_k(kt-t_1-t_2-\cdots-t_k) \quad (3)$$

and Stafford[11] assumed a "separable' form

$$\Psi_i(t_1, t_2, \cdots t_n) = f_i^{(1)}(t_1) f_i^{(2)}(t_2) \cdots f_i^{(n)}(t_n). \quad (4)$$

Both of these assumptions for the forms of the kernels are without any basis other than computational convenience and the studies employing them appear to exhibit poor results in terms of model verification with experiments.

III. Representation of an Elastomer Constitutive Relation

Consider an isotropic, homogeneous, and nonaging elastomer material under isothermal conditions. The linear constitutive equation for the viscoelastic material can be written as[14]

$$\sigma(t) = \int_{-\infty}^{t} g_1(t-t_1)\dot{\varepsilon}(t_1)dt_1, \quad (5)$$

where the function $g_1(t-t_1)$ is called the stress relaxation function and $\dot{\varepsilon}(t)$ is the strain history. The principle of superposition is valid in this linear model.

The general constitutive equation for nonlinear viscoelastic material was originally proposed by Green and Rivlin[15]. For the case of one-dimensional deformation the constitutive equation is given as[14]

$$\begin{aligned}\sigma(t) &= \int_{-\infty}^{t} g_1(t-t_1)\dot{\varepsilon}(t_1)dt_1 \\ &+ \int_{-\infty}^{t}\int_{-\infty}^{t} g_2(t-t_1, t-t_2)\dot{\varepsilon}(t_1)\dot{\varepsilon}(t_2)dt_1dt_2 \\ &+ \int_{-\infty}^{t}\int_{-\infty}^{t}\int_{-\infty}^{t} g_3(t-t_1, t-t_2, t-t_3) \\ &\quad \dot{\varepsilon}(t_1)\dot{\varepsilon}(t_2)\dot{\varepsilon}(t_3)dt_1dt_2dt_3 + \cdots, \quad (6)\end{aligned}$$

where the integrating functions $g_1(t-t_1)$, $g_2(t-t_1,t-t_2)$ and $g_3(t-t_1,t-t_2,t-t_3)$ are called the stress relaxation functions. This is a Volterra series representation between the stress and the strain rate. If there exits only one stress relaxation function g_1, then the material exhibits linear behavior. In the case of nonlinear material, higher order stress relaxation functions, i.e. higher order Volterra kernals, are additionally needed to represent the input-output relation. Thus, the above relation may be a general representation which describes nonlinear viscoelastic behavior.

IV. Modeling and Analysis of an Elastomer Force-Displacement Relation

We now consider an estimation of stress relaxation function due to pure step inputs by using the experimental data from the Instron Tester on the UPJOHN'S Urethane Elastomer sample. Since it is practically impossible to apply step inputs that have infinite slope at zero time, we use an alternative approach. Suppose that we apply a combined input, i.e. constant strain rate followed by a constant strain input shown in Fig.1. The basic assumption is that the estimated stress, $\sigma(t)$, due to a stress applied stepwise at zero time converges to the response due to a constant strain rate and constant strain input for $t > qt^*$ [16,17].

Since experimental data from the Instron Tester are obtained in terms of force displacement, we use the force-displacement relation instead of stress-strain relation. Eqn(6) can be rewritten as

$$\begin{aligned}f(t) &= \int_0^t l_1(t-t_1)\dot{x}(t_1)dt_1 \\ &+ \int_0^t\int_0^t l_2(t-t_1, t-t_2)\dot{x}(t_1)\dot{x}(t_2)dt_1dt_2 \\ &+ \int_0^t\int_0^t\int_0^t l_3(t-t_1, t-t_2, t-t_3) \\ &\quad \dot{x}(t_1)\dot{x}(t_2)\dot{x}(t_3) + \cdots, \quad (7)\end{aligned}$$

where all initial times are assumed to be 0 (zero) and l_1, l_2, and l_3 are Volterra kernels characterizing the input-output relation.

Eight indentation test inputs for the Urethane elastomer sample with a 0.9525 cm radius penetrator are shown in Fig.2. We use log f(t)-log t plot in order to fit the estimated f(t) to displacement applied stepwise at zero time, because it

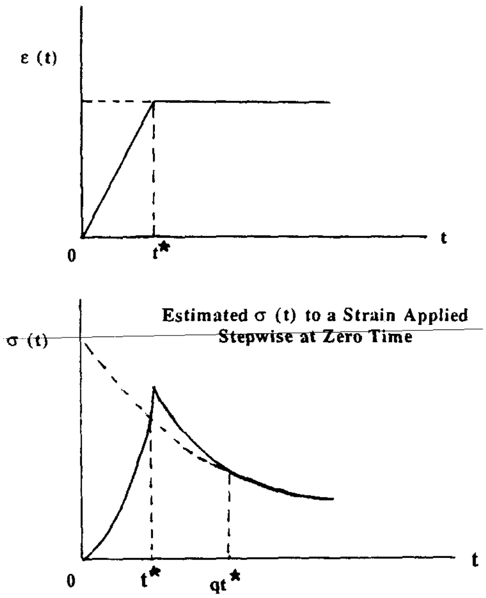

Fig.1. Typical strain input and stress output.

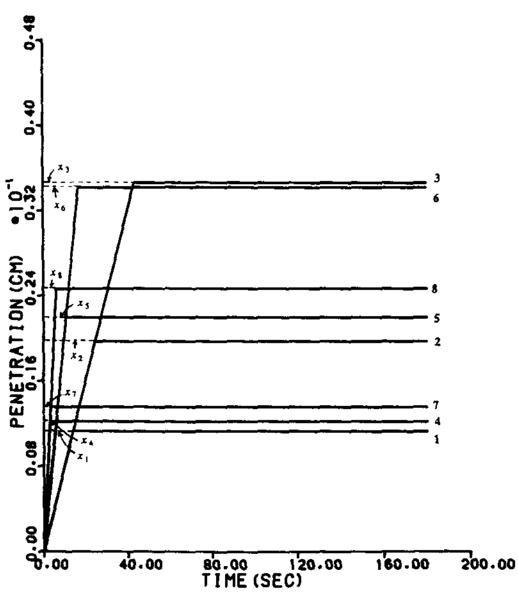

Fig.2. Indentation test inputs.

is easy to figure out the trend of curve. One log f(t)-log t plot is shown in Fig.3. From this plot, we can obtain the estimated relation.

$$\log f(t) = -0.089 \log t + 2.368. \tag{8}$$

The above equation is rewritten as

$$f(t) = 233.44 t^{-0.089}. \tag{9}$$

Similarily, we can estimate other forces due to diffierent displacements applied stepwise at zero time.

We now consider the second order truncated Volterra series model.

$$f(t) = \int_0^t h_1(t-t_1)\dot{x}(t_1)dt_1 + \int_0^t \int_0^t h_2(t-t_1, t-t_2)\dot{x}(t_1)\dot{x}(t_2)dt_1 dt_2, \tag{10}$$

where h_1 and h_2 are the first and second order Volterra kernels, respectively. Suppose that we have two input-output sets (force-displacement), $f_1(t)$-$x_1(t)$ and $f_2(t)$-$x_2(t)$. Let

$$x_1(t) = a_1 u(t) \tag{11}$$

$$x_2(t) = a_2 u(t), \tag{12}$$

where u(t) is a step function. Substituting $x_1(t)$ and $x_2(t)$ into eqn (10), and solving two algebraic equations for $h_1(t)$ and $h_2(t,t)$, we can obtain two Volterra kernels,

$$h_1(t) = \frac{a_2^2 f_2(t) - a_1^2 f_2(t)}{a_1 a_2 (a_2 - a_1)} \tag{13}$$

$$h_2(t, t) = \frac{a_1 f_2(t) - a_2 f_1(t)}{a_1 a_2 (a_2 - a_1)} \tag{14}$$

For the second order truncated Volterra series form, we use two deata sets, i.e. $f_1(t)$ and $f_3(t)$, and estimate $h_1(t)$ and $h_3(t,t)$. In order to verify these two kernels, we apply two step inputs, $x_2(t)$ and $x_8(t)$ shown in Fig.2 and estimated outputs, $\hat{f}_2(t)$ and $\hat{f}_8(t)$. These estimated outputs calculated using the kernels $h_1(t)$ and $h_2(t,t)$, and the corresponding kernels are shown in Fig.4 and Fig.5, respectively. The maximum error here is about 95.%.

We now extend the previous model to the third order truncated Volterra series model with the following form:

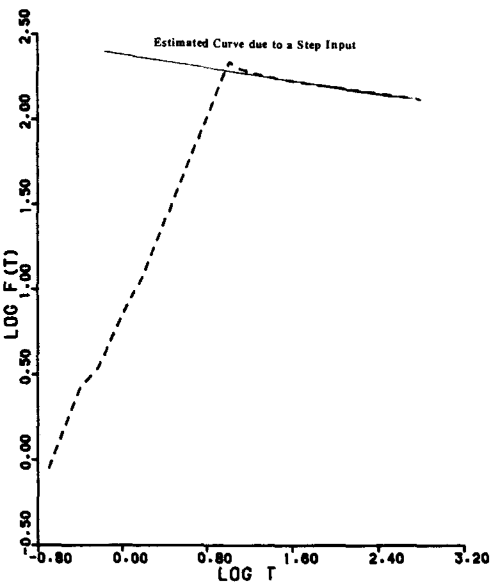

Fig.3. Log f(t)-log t plot.

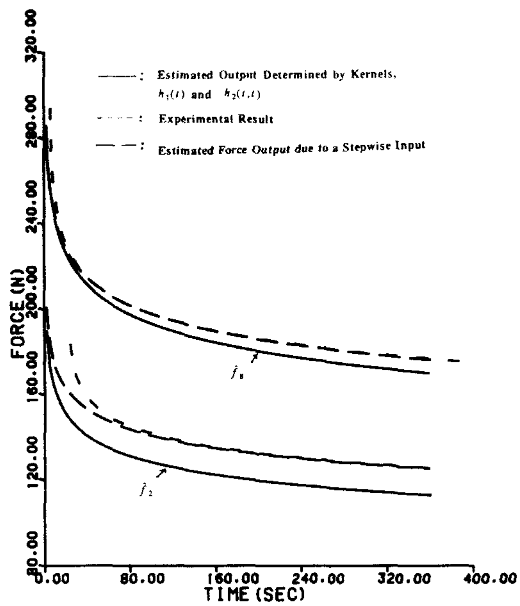

Fig.4. Estimated outputs of $f_2(t)$ and $f_8(t)$ from $f_1(t)$-$x_1(t)$ and $f_3(t)$-$x_3(t)$.

$$
\begin{aligned}
f(t) = & \int_0^t h_1(t-t_1)\dot{x}(t_1)dt_1 \\
& + \int_0^t \int_0^t h_2(t-t_1, t-t_2)\dot{x}(t_1)\dot{x}(t_2)dt_1dt_2 \\
& + \int_0^t \int_0^t \int_0^t h_3(t-t_1, t-t_2, t-t_3) \\
& \dot{x}(t_1)\dot{x}(t_2)\dot{x}(t_3)dt_1dt_2dt_3.
\end{aligned} \tag{15}
$$

Let

$$x_1(t) = a_1u(t) \tag{16}$$

$$x_2(t) = a_2u(t) \tag{17}$$

$$x_3(t) = a_3u(t), \tag{18}$$

and the corresponding force response be $f_1(t)$, $f_2(t)$, and $f_3(t)$, respectively. Then, substituting $x_1(t)$, $x_2(t)$ and $x_3(t)$ into eqn (15), and solving three algebraic equations for three kernels, we obtain

$$
\begin{aligned}
h_1(t) = & [f_1(t)(a_2^2a_3^3 - a_2^3a_3^2) - a_1^2(f_2(t)a_3^3 - f_3(t)a_2^3) \\
& + a_1^3(f_2(t)a_3^2 - f_3(t)a_2^2)]/\det
\end{aligned} \tag{19}
$$

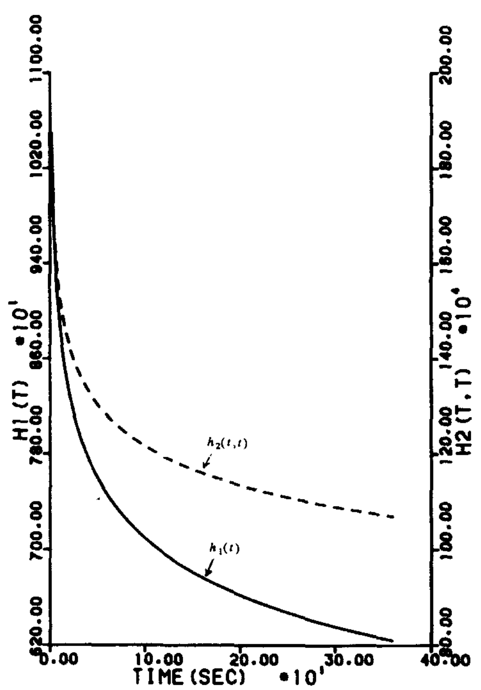

Fig.5. Estimated volterra kernels.

$$h_2(t,t)=[a_1(f_2(t)a_3^3-a_2^3f_3(t))-f_1(t)(a_2a_3^3-a_2^3a_3)+a_1^3(a_2f_3(t)-a_3f_2(t))]/\det \quad (20)$$

$$h_3(t,t,t)=[a_1(a_2^2f_3(t)-a_3^2f_2(t))-a_1^2(a_2f_3(t)-a_3f_2(t))+f_1(t)(a_2a_3^2-a_2^2a_3)]/\det, \quad (21)$$

where

$$\det=a_1a_2a_3(a_2a_3^2-a_2^2a_3-a_1a_3^2+a_1a_2^2+a_1^2a_3-a_1^2a_2).$$

For the third order truncated Volterra series model, we need three input-output data sets to estimate $h_1(t)$, $h_2(t,t)$ and $h_3(t,t,t)$. We assume that three data sets are $f_1(t)$-$x_1(t)$, $f_3(t)$-$x_3(t)$, and $f_5(t)$-$x_5(t)$. We now want to estimate $f_2(t)$, $f_6(t)$ and $f_8(t)$ due to step inputs, $x_2(t)$, $x_6(t)$, $x_7(t)$ and $x_8(t)$, respectively. Compared with the results for the second order truncated Volterra series model, the estimated outputs have better results as expected. The estimated outputs and kernels are shown in Fig.6 and Fig.7, respectively. The maximum error here is about 6.4%. Thus the third order Volterra series model has better results, compared with those of the second order Volterra series model.

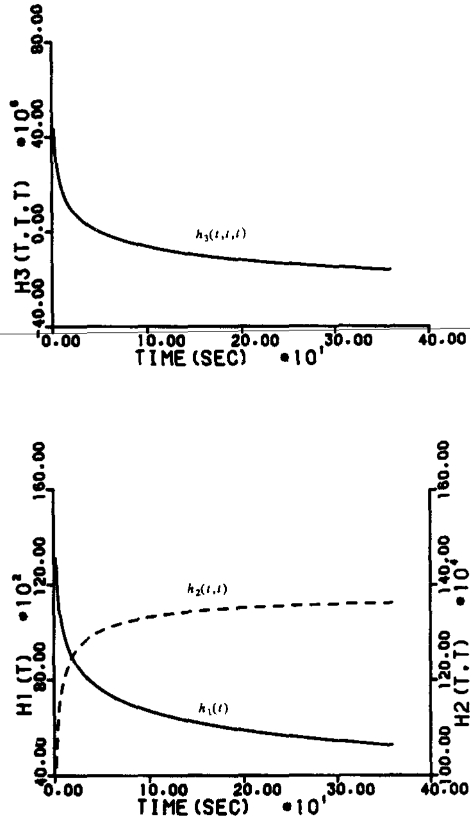

Fig.7. Estimated volterra kernels.

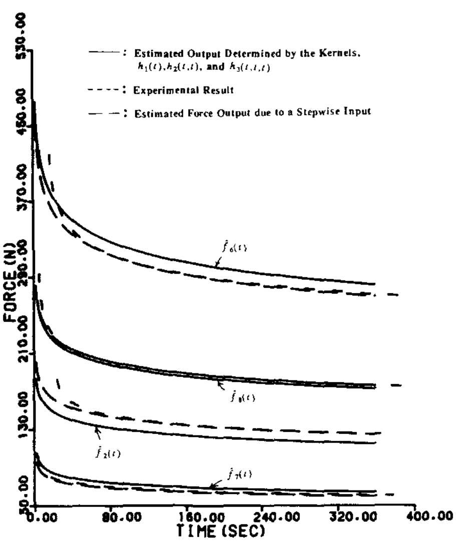

Fig.6. Estimated outputs form $f_1(t)$-$x_1(t)$, $f_3(t)$-$x_3(t)$, and $f_5(t)$-$x_5(t)$.

V. Conclusion

The input-output relations for nonlinear systems can be explicitly represented by the Volterra series and they can be characterized by the Volterra kernels. If the systems are not strongly nonlinear, they can be approximated by the truncated Volterra series solutions with only a few low order terms. The second order and third order truncated Volterra series models have been employed to estimate the force-displacement relation due to step inputs, which is one of the prominent properties to characterize the viscoelastic material. Actual experimental data from the Instron Tester are obtained for combined inputs, i.e. constant penetration rate followed by constant penetration inputs. These data are then estimated for step inputs and utilized for the truncated Volterra series models. The third order Volterra series model has better results, compared with those of the second order Volterra series model. This approach may be applied to modeling other elastomer constitutive relations including creep phenomena.

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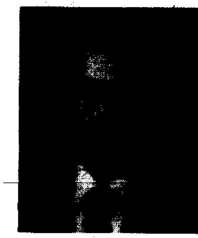

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