韓國數學教育學會誌<數學教育>
1989. 6, Vol. XXVIII, No. 1.

Journal of the Korea Society of Mathematical Education
Jun. 1989. Vol. XXVIII, No 1

On Eigenvalues of Continuous Function

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0. Introduction

In this paper we always assume that the phase spaces of the flows are compact metric spaces, unless specified otherwise. We introduce the eigenvalues of the flows and study the some properties.

Throughout this chapter, K denotes the unit circle in the complex plane.

1. Definitions and Notations

Definition 1.1: Let (X,T,Π) be a flow and let $h : T \to R$ be a given continuous group homomorphism. Then $\lambda \in R$ is called an *eigenvalue* of Π for h, if there exists a continuous $f : X \to K$ such that $f(\Pi(x,t)) = \exp(2\pi i\lambda h(t))f(x)$, for all $(x,t) \in X \times T$. Such f is called an *eigenfunction* of Π with respect to λ for h.

Example 1.2: Let $h : R \to R$ be an identity map. Let $X = R$, $\Pi : X \times R \to X$ by $\Pi(x,t) = x + t$, then (X,R,Π) is a flow. Let $f : X \to K$ by $f(x) = \exp(2\pi\lambda xi)$. Then f is a continous function and

$$\begin{aligned} f(\Pi(x,t)) = f(x+t) &= \exp(2\pi\lambda(x+t)i) \\ &= \exp(2\pi\lambda xi)\exp(2\pi\lambda ti) \\ &= \exp(2\pi i\lambda t)f(x). \end{aligned}$$

Therefore λ is an eigenvalue of Π for h and f is an eigenfunction of Π with respect to λ for h. We denote $\Lambda_h(\Pi)$ the set of eigenvalues of Π for h.

Before starting the main result, we need the lemmas and the definitions.

Definition 1.3: A homomorphism $P : X \to Y$ is said to be *of distal type*, if whenever $x_1, x_2 \in P^{-1}(y)$ and $x_1 \neq x_2$, then there is an $\alpha = \alpha(x_1,x_2) > 0$ such that $d(x_1t, x_2t) \geq \alpha$ for all $t \in T$.

Definition 1.4: If $p : X \to Y$ and $0 < N < \infty$ then X is said to be an *N-fold covering* of Y with covering projection p if card $p^{-1}(y) = N$ for all $y \in Y$ and for each $y \in Y$ there is an open neighborhood V of y such that $p^{-1}(V)$ consists of N disjoint open sets U_i and $(p|U_i) : U_i \to V$ is a homeoomorphism, $i = 1, \ldots, N$.

Definition 1.5: Let (X,Y) and (Y,T) be flow. There exists a homomorphism $h : (X,T) \to (Y,T)$ such that $h^{-1}(h(x)) = \{x\}$ for some point $x \in X$. Then (X,T) is said to be an *almost automorphism extension* of (Y,T).

Lemma 1.6. ([4], structure theorem). *Let (X,T) and (Y,T) be the transformation groups where X and Y are compact metric spaces. Assume (Y,T) is minimal, $p : X \to Y$ is a homomorphism. Then the following are pairwise equivalent:*

(A) *p is of distal type and there exists $y_0 \in Y$, $\operatorname{card} p^{-1}(y_0) = N$, $0 < N < \infty$.*
(B) *$\operatorname{card} p^{-1}(y) = N$, for $y \in Y$ for some N, $0 < N < \infty$.*
(C) *X is an N-fold covering of Y.*

Definition 1.7 [3]: Let (X, Y) be a flow. A point x in X is an almost automorphic point if given any net $\{t_{\alpha'}\}$, $\alpha' \in \Lambda'$, there exists a subnet $\{t_\alpha\}$, $\alpha \in \Lambda$, such that $\lim xt_\alpha = z$ and $\lim zt_\alpha^{-1} = x$. We say a minimal set is *almost automorphic* flow if it contains at least one almost automorphic point. We denote $A(\Pi)$ the set of almost automorphic point of (X, T, Π).

2. Resutls

Proposition 2.1. $\Lambda_h(\Pi)$ *is subgroup of* R.

Proof: Let $\lambda, \gamma \in \Lambda_h(\Pi)$. Then there exists continuous functions $f, g : X \to K$ such that

$$\exp(2\pi i\lambda h(t))f(x) = f(\pi(x,t))$$

and

$$\exp(2\pi i\gamma h(t))g(x) = g(\Pi(x,t)),$$

for any $(x,t) \in X \times T$ and so $\exp(2\pi i(\gamma - \lambda)h(t)g(x)/f(x) = g(\Pi(x,t))/f(\Pi(x,t))$, for any $(x,t) \in X \times T$. Now we define $F : X \to K$ by $F(x) = g(x)/f(x)$. Then $\exp(2\pi i(\gamma - \lambda)h(t)F(x) = F(\Pi(x,t))$, for any $(x,t) \in X \times T$. Therefore $\gamma - \lambda \in \Lambda_h(\Pi)$. This completes the proof.

Lemma 2.2. *Let (X,T) be a flow and let (Y,T) a minimal flow. Let $h : (X,T) \to (Y,T)$ be a homomorphism. Suppose that there exists $x_0 \in X$ such that $h^{-1}(h(x_0)) = \{x_0\}$. If h is of distal type. Then $h : (X,T) \to (Y,T)$ is an isomorphism.*

Proof: Since (Y,T) is minimal, h is surjection. Since h is of distal type, by Lemma 1.6 (A), (B), $h^{-1}(y)$ is single point, for all $y \in Y$. Therefore h is one-to-one. Hence $h : (X,T) \to (Y,T)$ is an isomorphism.

Lemma 2.3. *Let (X,T,Π), (Y,T,ρ) be flows. If (X,T) is an extension of (Y,T), then* $\Lambda_h(\rho) \subset \lambda_h(\Pi)$.

Proof: Let $\lambda \in \Lambda_h(\rho)$. Then there exists a continuous function $f : Y \to K$ such that

$$f(\rho(y,t)) = f(y)\exp(2\pi i\lambda h(t)),$$

for all $(y,t) \in Y \times T$. Let $\phi : (X,T) \to (Y,T)$ be a homomorphism. Then $f \circ \phi : X \to K$ is a continuous function and

$$\begin{aligned}(f \circ \phi)(\Pi(x,t)) &= f(\phi(\Pi(x,t)) \\ &= f(\rho(\phi(x),t)) \\ &= f(\phi(x))\exp(2\pi i\lambda h(t)),\end{aligned}$$

for all $(x,t) \in X \times T$. Hence $\lambda \in \Lambda_h(\Pi)$.

The following theorem gives us a condition which has the same eigenvalue sets.

Theorem 2.4. *Let (X,T,Π) and (Y,T,ρ) be minimal. If (X,T,Π) is an almost automorphic extention of (Y,T,ρ), then $\Lambda_h(\Pi) = \Lambda_h(\rho)$.*

Proof: Let $\phi : (X,T) \to (Y,T)$ be a homomorphism such that $\phi^{-1}(\phi(x)) = \{x\}$ for some point $x \in X$. Let $\lambda \in \Lambda_h(\pi)$. Then there exists a continuous function $f : X \to K$ such that

$$f(\Pi(x,t)) = f(x)\exp(2\pi i\lambda h(t)), \quad \text{for all} \quad (x,t) \in X \times T.$$

Let $\rho^\lambda : (Y \times K) \times T \to (Y \times K)$ by $\rho^\lambda((y,z),t) = (\rho(t,t), z\exp(2\pi ih(t)))$, for all $(y,t) \in Y \times K$, and for all $t \in T$. Then ρ^λ is clearly continuous and

$$\rho^\lambda((y,z),e) = (\rho(y,e), z\exp(2\pi ih(e))) = (y,z).$$

$$\begin{aligned}\rho^\lambda((y,z),st) =& (\rho(y,st), z\exp(2\pi ih(st)))\\ =& (\rho(\rho(y,s),t)), z\exp(2\pi ih(s))\exp(2\pi ih(t))\rho^\lambda(\rho^\lambda(y,z),s),t))\\ =& \rho^\lambda((\rho(y,s), z\exp(2\pi ih(s)),t)\\ =& \rho((\rho(y,s),t), z\exp(2\pi ih(s))\exp(2\pi ih(t))).\end{aligned}$$

That is, $\rho^\lambda((y,z),st) = \rho^\lambda(\rho^\lambda(y,z),s),t)$. Therefore $(Y \times K, T, \rho^\lambda)$ is a flow. Let $H : X \to Y \times K$ by $H(x) = (\phi(x), f(x))$, for all $x \in X$. Then

$$\begin{aligned}H(\Pi(x,t)) &= (\phi(\Pi(x,t)), f(\Pi(x,t)))\\ &= (\rho(\phi(x),t), f(x)\exp(2\pi i\lambda h(t))\\ &= \rho^\lambda((\phi(x), f(x)),t)\\ &= \rho^\lambda(H(X),t), \quad \text{for all} \quad (x,t) \in X \times T.\end{aligned}$$

Therefore $H : (X,T,\Pi) \to (Y \times K, T, \rho^\lambda)$ is homomorphism. Let $P_Y : Y \times K \to Y$ be the projection. Then $P_Y : (Y \times K, T, \rho^\lambda) \to (Y,T,\rho)$ is a homomorphism. Indeed,

$$\begin{aligned}P_Y(\rho^\lambda((y,z),t) &= P_Y(\rho(y,t), z\exp(2\pi i\lambda h(t))\\ &= \rho(y,t).\end{aligned}$$

Let $M = H(X)$. Then M is an invariant set of $(Y \times K, T, \rho^\lambda)$. For, let $y \in M$, there is $x \in X$ such that $y = H(x)$ and

$$\begin{aligned}\rho^\lambda(y,t) &= \rho^\lambda(H(x),t)\\ &= \rho^\lambda((\phi(x), f(x)),t)\\ &= (\rho(\phi(x),t), f(x)\exp(2\pi i\lambda h(t)))\\ &= (\phi(\Pi(x,t)), f(x)\exp(2\pi i\lambda h(t)))\\ &= (\phi(\Pi(x,t)), f(\Pi(x,t))) \subset (\phi(X), f(x)).\end{aligned}$$

Clearly, $\phi(x) = P_Y \circ H(x)$ for all $x \in X$. Let $\phi^{-1}(y_0) = x_0$. Take $m_1, m_2 \in P_y^{-1}(y_0) \cap M$, $x_1 \in H^{-1}(m_1)$, $x_2 \in H^{-1}(m_2)$. Then

$$\phi(x_1) = P_Y \circ H(x_1) = P_Y(m_1) = y_0,$$
$$\phi(x_2) = P_Y \circ H(x_2) = P_Y(m_2) = y_0.$$

Hence $x_1, x_2 \in \phi^{-1}(y_0)$. Since $\phi^{-1}(y_0) = \{x_0\}$, we have $x_1 = x_2 = x_0$ and hence $m_1 = H(x_1) = H(x_2) = m_2$. Therefore $P_Y^{-1}(y_0) \cap M$ is a single point and $P_Y|_M : (M, T, \rho^\lambda|_M) \to (Y, T, \rho)$ is a homomorphism. Clearly, $P_Y|_M$ is distal type, by Lemma 2.2., $P_Y|_M$ is an isomorphism. Hence $\Lambda_h(\rho) = \Lambda_h(\rho^\lambda|_M)$. Let $P_K : Y \times K \to K$ be the projection. Then

$$\begin{aligned} P_K(\rho^\lambda((y,z),t)) &= P_K(\rho(y,t), z\exp(2\pi i\lambda h(t))) \\ &= z\exp(2\pi i\lambda h(t)) \\ &= P_K(y,z)\exp(2\pi i\lambda h(t)), \quad \text{for all} \end{aligned}$$

$(y,z) \in M$, for all $t \in T$. That is, P_K is an eigenfunction of $\rho^\lambda|_M$ with respect to λ for h. This implies $\lambda \in \Lambda_h(\rho^\lambda|_M) = \Lambda_h(\rho)$. We have proved that $\Lambda_h(\Pi) \subset \Lambda_h(\rho)$. By the previous Lemma 2.3., $\Lambda_h(\rho) \subset \Lambda_h(\Pi)$. Hence we obtain the relation $\Lambda_h(\Pi) = \Lambda_h(\rho)$.

Theorem 2.5. *Let (X,T,Π) and (Y,T,ρ) be the minimal flows. Let $h : X \to Y$ be a homomorphism. If (X,T,Π) is an almost automorphic flows. Then $h(A(\Pi)) \subset A(\rho)$.*

Proof: Let $y \in h(A(\Pi))$ and let $\{t_\alpha\}$ be a net in T. Then there exists $x \in A(\Pi)$ and a subnet $\{t_{\alpha'}\}$ of $\{t_\alpha\}$ such that $y = h(x)$ and

$$\begin{aligned} \Pi(x, t_{\alpha'}) &\longrightarrow x_0 \in X \quad \text{and} \\ \Pi(x_0, t_{\alpha'}^{-1}) &\longrightarrow x \in X. \end{aligned}$$

Since

$$\begin{aligned} h(\Pi(x, t_{\alpha'})) &= \rho(h(x), t_{\alpha'}) \\ &= \rho(y, t_{\alpha'}) \longrightarrow h(x_0), \\ h(\Pi(x_0, t_{\alpha'}^{-1})) &= \rho(h(x_0), t_{\alpha'}^{-1}) \longrightarrow h(x) = y. \end{aligned}$$

Therefore $y \in A(\rho)$. Hence $h(A(\Pi)) \subset A(\rho)$.

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