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Proper Efficiency and Zero-likeness of Multiobjective Programming with Set Functions

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ABSTRACT. Subdifferentiability of a vector-valued set function is defined first. Then in terms of zero-like functions, a perperly efficient solution of a convex programming problem with set functions will be characterized.

1. Preliminaries

For a subset A in R^n, its (strictly) positive polar $A^0(A^{so})$ is defined by

$$A^c(A^{so}) = \{x^* \in R^n : \langle x, x^* \rangle \geq (>) 0 \quad \text{for any} \quad x \in A\}.$$

Let Y be an objective space in R^n. An efficient point of Y with respect to the domination structure $D \subseteq R^n$ is the point $y^* \in Y(y \in \text{Eff}(Y, D))$ such that

$$(Y - y^*) \cap (-D) = \{0\}, \quad \text{or} \quad \phi \quad \text{if} \quad 0 \notin D.$$

That is, there is no $y \neq y^* \in Y$ such that $y^* \in y + D$.

Another restricted solution concept, proper efficiency, eliminates efficient points of certain types of abnormality. A point $y^* \in Y$ is called a properly efficient point of $Y(y^* \in P(Y, D))$ with respect to a domination cone D [1] if

$$cl(P(Y + D - y^*)) \cap (-D) = \{0\}, \quad \text{or} \quad \phi \quad \text{if} \quad 0 \notin D,$$

where $P(S) = \{\alpha y : \alpha > 0, y \in S\}$ is the projecting cone for a set $S \subset R^n$. Let K be a cone of R^n and $x, y \in R^n$. Then we denote

(i) $x <_K y$ if and only if $y - x \in \text{int}\, K$
(ii) $x \leq_K y$ if and only if $y - x \in K \backslash \{0\}$
(iii) $x \leq_K y$ if and only if $y - x \in K$.

Let $(X, \mathfrak{A}, m)$ be a measure space where $\mathfrak{A}$ is the σ-algebra of all m-measurable subsets of X. For $\Omega \in \mathfrak{A}$, χ_Ω denotes the characteristic function of Ω. We shall write L_p instead of $L_p(X, \mathfrak{A}, m)$ for $0 < p \leq \infty$.

If $(X, \mathfrak{A}, m)$ is finite, atomless and L_1 is separable, then for any $\Omega_1, \Omega_2 \in \mathfrak{A}$ and $\alpha \in I = [0, 1]$, there exists a sequence $\{\Gamma_n\} \subset \mathfrak{A}$ such that

$$\chi_{\Gamma_n} \xrightarrow{w^*} \alpha \chi_{\Omega_1} + (1 - \alpha) \chi_{\Omega_2},$$

where $\xrightarrow{w^*}$ denotes the weak* convergence of elements in L_∞ [6]. Such a sequence $\{\Gamma_n\}$ is called a Morris sequence associated with $(\alpha, \Omega_1, \Omega_2)$.

Definition 1.1: [2] A subfamily $\mathfrak{S}$ of $\mathfrak{A}$ is called convex if, given $(\alpha, \Omega_1, \Omega_2) \in [0,1] \times \mathfrak{S} \times \mathfrak{S}$ and a Morris-sequence $\{\Gamma_n\}$ in $\mathfrak{A}$ associated with $(\alpha, \Omega_1, \Omega_2)$, there exists a subsequence $\{\Gamma_{n_k}\}$ of $\{\Gamma_n\}$ in $\mathfrak{S}$.

Definition 1.2: [2] Let $\mathfrak{S}$ be a convex subfamily of $\mathfrak{A}$. Let K be a convex cone of R^n. A multi-valued set function $H : \mathfrak{S} \to R^n$ is called K-convex if, given $(\alpha, \Omega_1, \Omega_2)$ in $I \times \mathfrak{S} \times \mathfrak{S}$ and any Morris-sequence $\{\Gamma_n\}$ in $\mathfrak{A}$ associated with $(\alpha, \Omega_1, \Omega_2)$, there exists a subsequence $\{\Gamma_{n_k}\}$ of $\{\Gamma_n\}$ in $\mathfrak{S}$ such that

$$\limsup_{k \to \infty} H(\Gamma_{n_k}) \leq_K \alpha H(\Omega_1) + (1-\alpha) H(\Omega_2),$$

where lim sup is taken over each component.

If $\mathfrak{S}$ is convex, then $Cl(\mathfrak{S})$, the weak*-closure of $\mathfrak{S}$ *in* L_∞ is the w^*-closed convex hull of $\mathfrak{S}$ [3].

Definition 1.3: [3] A vector-valued set function $H = (H_1, \ldots, H_n) : \mathfrak{S} \to R^n$ is called w^*-continuous on $\mathfrak{S}$ if for each $f \in cl(\mathfrak{S})$ and for each $j = 1, \ldots, n$, $\{H_j(\Omega_m)\}$ converges to the same limit for all $\{\Omega_m\}$ with $\chi_{\Omega_m} \xrightarrow{w^*} f$.

2. Multiobjective Programming Problem with Set Functions

Multicbjective programming problem with set functions can be described as follows:

$$\text{(P)} \qquad \begin{aligned} &\text{Min}_D\, F(\Omega) \\ &\text{subject to} \quad \Omega \in \mathfrak{S}, \\ &\text{and} \quad G(\Omega) \leq_Q 0, \end{aligned}$$

which has been defined as the problem of finding all feasible efficient or properly efficient points of $F(\mathfrak{S})$ with respect to the domination cone D. That is, letting $\mathfrak{S} = \{\Omega \in \mathfrak{S} : G(\Omega) \leq_Q 0\}$, we want to find $\Omega^* \in \mathfrak{S}$ such that $F(\Omega^*) \in \mathrm{Eff}(F(\mathfrak{S}'), D)$ or $F(\Omega^*) \in \mathfrak{P}(F(\mathfrak{S}'), D)$ in the problem (P). We shall call such $\Omega^* \in \mathfrak{S}$ an efficient D-solution or properly efficient D-solution to the problem (P), respectively.

A necessary condition for a properly efficient D-solution to the problem (P) is proven through associated scalar problem. Results obtained in this section are analogous to those obtained by Hsia and Lee in [5].

The following assumptions are imposed on the problem (P):

(i) D and Q are pointed closed convex cones in R^p and R^m with nonempty interiors, respectively,

(ii) $F : \mathfrak{S} \to R^p$ is D-convex, w^*-continuous,

(iii) $G : \mathfrak{S} \to R^m$ is Q-convex, w^*-continuous,

(iv) $\mathfrak{S}$ is a convex subfamily of the σ-algebra $\mathfrak{A}$.

The next lemma relates properly efficient points with scalarized programming problems.

Lemma 2.1. *Let Y be a D-convex set of R^p and D a closed and pointed convex cone in R^p. Then $y^* \in \mathfrak{P}(Y, D)$ if and only if there exists $\mu^* \in \operatorname{int} D^0$ such that*

$$\langle \mu^*, y^* \rangle \leq \langle \mu^*, y \rangle \quad \textit{for all} \quad y \in Y.$$

Proof: Note that $\operatorname{int} D^0 \neq \phi$ and $\operatorname{int} D^0 = D^{s0}$. Then, by [8, Theorems 3.4.1, 3.4.2, 3.4.3],

$$P(Y, D) = \bigcup_{\mu \in D^{s0}} \{y^* \in Y : \langle \mu, y^* \rangle = \inf\{\langle \mu, y \rangle : y \in Y\}\}.$$

Thus we obtain the conclusion.

Next we give the Lagrange multiplier theorem for vector-valued programming with set functions. It is a generalization of Theorem 3.1 [3]. For the cones D and Q, the set of $p \times m$ matrices $\{M \in R^{p \times m} : MQ \subset D\}$ is denoted by $\mathfrak{L}$. Such matrices are called positive in some literature [8].

Proposition 2.2. *Let Ω^* be a properly efficient D-solution to the problem (P). If there is $\Omega_0 \in \mathfrak{S}$ such that $G(\Omega_0) <_Q 0$, then there exists $M^* \in \mathfrak{L}$ such that*

(i) $F(\Omega^*) \in \operatorname{Min}_D\{L(\Omega, M^*) : \Omega \in \mathfrak{S}\}$
(ii) $M^* G(\Omega^*) = 0$,

where $L(\Omega, M) = F(\Omega) + MG(\Omega)$ for $\Omega \in \mathfrak{S}$ and $M \in R^{p \times m}$.

Proof: [5]

The multiplier M^* obtained in Proposition 2.2 is called *Lagrange multiplier.*

3. Proper Efficiency and Zero-likeness

In this section, the preperly efficient D-solution of the multiobjective programming problem (P) introduced in the previous section is characterized in terms of zero-like functions. For this purpose, let the domination cone D be the nonnegative orthant R^p_+ so that $\operatorname{int} D^0 = \operatorname{int} R^p_+$. Also let $Q = R^m_+$. We denote the dual space of L_∞ with the norm topology by $(L_\infty)^*$, which can be characterized as the space of finitely additive set functions [10]. We shall use the functional notation $\langle f, \chi_\Omega \rangle = \int_\Omega f \, dm$. First we extend the notion of subgradient given in [9] to the case of vector-valued set functions.

Definition 3.1: A vector-valued set function $F : \mathfrak{A} \to R^p$ is called subdifferentiable at $\Omega_0 \in \mathfrak{A}$ if there exists an element

$$T = (T_i)_{i=1}^p \in X_{i=1}^p (L_\infty)^* = (L_\infty)^* \times \cdots \times (L_\infty)^* \quad p-\text{times},$$

called the *subgradient* of F at Ω_0, such that

$$F(\Omega) \geq_0 F(\Omega_0) + \ll T, \chi_\Omega - \chi_{\Omega_0} \gg \quad \text{for all } \Omega \in \mathfrak{A},$$

where $\ll T, \chi_\Omega \gg = (\langle T_1, \chi_\Omega \rangle, \langle T_2, \chi_\Omega \rangle, \ldots, \langle T_p, \chi_\Omega \rangle)^t$.

The set of all subgradients of F at Ω_0 is called the *subdifferential* of F at Ω_0 and is denoted by $\partial F(\Omega_0)$. Let us denote $\overline{R}^p = R^p \cup \{\pm\infty\}$.

Now we define a zero-like function on $X_{i=1}^p (L_\infty)^*$.

Definition 3.2: An element $T = (T_i)_{i=1}^p \in X_{i=1}^p (L_\infty)^*$ is called *zero-like*, written as $T \sim 0$ or $0 \sim T$, if there exists positive numbers $\mu_1, \ldots, \mu_p$ such that

$$\mu_1 T_1 + \cdots + \mu_p T_p = 0$$

as a function in $(L_\infty)^*$.

In terms of zero-like functions, a properly effificient solution of a convex programming can be characterized. First we consider the unconstraint programming problem:

$$\text{(P}'\text{)}\qquad \begin{array}{c}\operatorname{Min} F(\Omega)\\ \text{subject to } \Omega \in \mathfrak{S},\end{array}$$

where $F = (F_i)_{i=1}^p : \mathfrak{S} \to \overline{R}^p$ is w^*-continuous and $\mathfrak{S}$ is a convex subfamily. Suppose that $\operatorname{dom} F = \bigcap_{i=1}^p (\operatorname{dom} F_i) = \mathfrak{S}$ and that $cl(\mathfrak{S})$ contains relative interior points.

Theorem 3.3. *For the problem (P′), Ω^* is a properly efficient solution if and only if there is a zero-like T in $\partial F(\Omega^*) = \partial F_{\mathfrak{S}}(\Omega^*)$.*

Proof: Suppose that there is a zero-like T in $\partial F(\Omega^*)$. Then there exists a $\mu \in \operatorname{int} R_+^p$ such that $\langle \mu, T\rangle = 0$. Since $T \in \partial F(\Omega^*)$, $F(\Omega) \geq F(\Omega^*) + \ll T, \chi_\Omega - \chi_{\Omega^*} \gg$ for all $\Omega \in \mathfrak{S}$. Applying μ on both sides, we have

$$\begin{aligned}\langle \mu, F(\Omega)\rangle &\geq \langle \mu, F(\Omega^*)\rangle + \langle \mu, \ll T, \chi_\Omega - \chi_{\Omega^*} \gg\rangle\\ &= \langle \mu, F(\Omega^*)\rangle + \langle \langle \mu, T\rangle, \chi_\Omega - \chi_{\Omega^*}\rangle\\ &= \langle \mu, f(\Omega^*)\rangle, \quad \text{for all} \quad \Omega \in \mathfrak{S}.\end{aligned}$$

Thus $\langle \mu, F(\Omega^*)\rangle \in \operatorname{Eff}(\langle \mu, F(\Omega)\rangle, R_+^p)$. Then, by Lemma 2.1, Ω^* is a properly efficient solution. Conversely, let Ω^* be a properly efficient solution. Then, by Lemma 2.1, there exists a $\mu \in \operatorname{int} R_+^p$ such that

$$\langle \mu, F(\Omega^*)\rangle \leq \langle \mu, F(\Omega)\rangle, \quad \text{for all} \quad \Omega \in \mathfrak{S}.$$

Clearly, $0 \in \partial(\langle \mu, F(\Omega^*)\rangle)(\Omega^*)$. Writing $F(\Omega) = (F_1(\Omega), \ldots, F_p(\Omega))$, we have

$$0 \in \partial\Big(\sum_{i=1}^p \mu_i \cdot F_i\Big)(\Omega^*) = \sum_{i=1}^p \mu_i \cdot \partial F(\Omega^*).$$

Therefore, there are $T_i \in \partial F_i(\Omega^*)$, $i = 1, \ldots, p$, such that

$$0 = \sum_{i=1}^p \mu_i \cdot T_i.$$

Then $T = (T_i) \in X_{i=1}^p (L_\infty)^*$ and $T \sim 0$ by μ. Hence, for each $i = 1, \ldots, p$,

$$F_i(\Omega) \geq F_i(\Omega^*) + \langle T_i, \chi_\Omega - \chi_{\Omega^*}\rangle \quad \text{for all} \quad \Omega \in \mathfrak{S}.$$

Therefore, $T \in \partial F(\Omega^*)$. The proof is now complete.

Next we consider a sufficient and necessary condition for the existence of a properly efficient solution of a multiobjective programming problem with constraints:

$$\text{(P)}\qquad \begin{array}{c}\operatorname{Min} F(\Omega)\\ \text{subject to } \Omega \in \mathfrak{S}\\ \text{and} \quad G(\Omega) \leq_Q 0.\end{array}$$

Theorem 3.4. *Let $\mathfrak{S} = \mathfrak{S} \cap \operatorname{dom} F \cap \operatorname{dom} G$ and $cl(\mathfrak{S})$ have nonempty relative interior. Suppose that F and G are w^*-continuous on $\mathfrak{S}$. Assume Slater's constraint qualification holds. That is, there is $\Omega_0 \in \mathfrak{S}$ such that $F(\Omega_0) < 0$. Then an $\Omega^* \in \mathfrak{S}$ is a properly efficient solution of* (P) *if and only if there exists a zero-like $T \in X_{i=1}^{p\times m}(L_\infty)^*$ such that*

$$T = (S_1, S_2) \text{ with } S_1 \in \partial F(\Omega^*) \text{ and } S_2 \in \partial G(\Omega^*).$$

Moreover, if $S_1 \in \partial F(\Omega^*)$ is zero-like, then Ω^* is a properly efficient solution to the problem (P).

Proof: If $F(\Omega^*)$ is a properly efficient point for the problem (P), then there is a $\mu \in \operatorname{int} R_+^p$ and $\lambda \in Q^0$ such that

$$\langle \mu, F(\Omega^*) \rangle = \operatorname{Min}\{\langle \mu, F(\Omega)\rangle + \langle \lambda, G(\Omega)\rangle : \Omega \in \mathfrak{S}\},$$

by the Lagrange multiplier Theorem 2.2. Therefore,

$$\begin{aligned} 0 &\in \partial(\langle \mu, F(\Omega^*)\rangle + \langle \lambda, G(\Omega^*)\rangle) \\ &= \partial(\langle \mu, F(\Omega)\rangle) + \partial(\langle \mu, G(\Omega^*)\rangle). \end{aligned}$$

Thus, there is an $S_1 \in \partial F(\Omega^*)$ and an $S_2 \in \partial G(\Omega^*)$ such that

$$0 = \langle \mu, S_1 \rangle + \langle \lambda, S_2 \rangle$$

Let $T = (S_1, S_2)$. The converse is immediate from the definition and Lemma 2.1. The proof is therefore complete.

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