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Consistency Theorem for the Integrals of Riemann-Stieltjes and Lebesgue Types

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0. Introduction

In [**5**, Theorem 11.33] the consistency theorem says that every function which is Riemann integrable on an interval is also Lebesgue integrable and their integrals are consistent.

The purpose of this paper is to give an elementary proof of the consistency theorem for the integrals of Riemann-Stieltjes and Lebesgue types using the standard proof of Theorem 11.33 (a) in [**5**]. For this we use Ross' definition [**2**] of the Riemann-Stieltjes integrals.

Our main references for the basic notions about Lebesgue measure theory are [**1**] and [**4**].

1. Usual Definition of the Riemann-Stieltjes Integral

Let $\alpha : [a,b] \to \mathbf{R}$ be a monotonically increasing function. For any partition

$$P = \{a = x_0 \le x_1 \le \cdots \le x_n = b\}$$

of $[a,b]$, we write

$$\Delta\alpha_i = \alpha(x_i) - \alpha(x_{i-1}), \qquad i = 1, 2, \ldots, n.$$

Suppose $f : [a,b] \to \mathbf{R}$ is a bounded function and define the numbers

$$\text{(1)} \qquad \begin{aligned} M_i &= \sup\{f(x) : x_{i-1} \le x \le x_i\} \\ m_i &= \inf\{f(x) : x_{i-1} \le x \le x_i\}, \qquad i = 1, 2, \ldots, n. \end{aligned}$$

The upper and lower Riemann-Stieltjes sums associated with P are defined as

$$\text{(2)} \qquad U(P,f,\alpha) = \sum_{i=1}^{n} M_i \Delta\alpha_i \quad \text{and} \quad L(P,f,\alpha) = \sum_{i=1}^{n} m_i \Delta\alpha_i.$$

Now the upper and lower Riemann-Stieltjes integrals of f with respect to α, over $[a,b]$ are defined as

$$\text{(3)} \qquad \overline{\int_a^b} f\,d\alpha = \inf_P U\{P,f,\alpha) \quad \text{and} \quad \underline{\int_a^b} f\,d\alpha = \sup_P L(P,f,\alpha),$$

where the inf and sup being taken over all partitions. We denote the common value by

$$\text{(4)} \qquad \int_a^b f\,d\alpha$$

if the upper and lower Riemann-Stieltjes integrals (3) are equal. This is the *Riemann-Stieltjes integral* (or simply the *Stieltjes integral*) of f with respect to α, over $[a,b]$. If (4) exists, we say that f is *Riemann-Stieltjes integrable* with respect to α and write $f \in \mathfrak{R}(\alpha)$ on $[a,b]$. Also we write $f \in \mathfrak{R}$ on $[a,b]$ when f is Riemann integrable on $[a,b]$.

By taking $\alpha(x) = x$, the Riemann integral is seen to be a special case of the Riemann-Stieltjes integral.

Example 1: If we consider the functions defined by

$$f(x) = \begin{cases} 0 & \text{if } x = 0 \\ 1/x & \text{if } 0 < x < 1 \\ 1 & \text{if } 1 \le x \le 2 \end{cases}$$

and

$$\alpha(x) = \begin{cases} 1 & \text{if } 0 \le x \le 1 \\ x & \text{if } 1 < x \le 2, \end{cases}$$

then $\int_0^2 f(x)\,dx$ does not exist. However $\int_0^2 f\,d\alpha = 1$.

Moreover, the Riemann-Stieltjes integral of f may exist when α need not even consinuous : Let $f(x) = 1$ on $[0,1]$ and

$$\alpha(x) = \begin{cases} 0 & \text{if } 0 \le x \le 1/2 \\ 1 & \text{if } 1/2 \le x \le 1. \end{cases}$$

Clearly α is not continuous at $x = 1/2$. Note that $\int_0^1 f\,d\alpha = 1$.

2. Ross' Definition of the Riemann-Stieltjes Integral

Ross [2] proposed a new definition which is a generalization of the standard Darboux definition explained in the previous section.

First we need some notations. For the monotonically increasing function α defined on $[a,b]$, we write

$$\alpha(x^+) = \lim_{t \to x^+} \alpha(t) = \sup_{a<t<x} \alpha(t) \tag{5}$$
$$\alpha(x^-) = \lim_{t \to x^-} \alpha(t) = \inf_{x<t<b} \alpha(t).$$

For the endpoints we decree

$$\alpha(a^-) = \alpha(a) \quad \text{and} \quad \alpha(b^+) = \alpha(b).$$

Note that

$$\alpha(x^-) \le \alpha(x^+), \qquad a \le t \le b.$$

If α is continuous at x, then we have

$$\alpha(x^-) = \alpha(x) = \alpha(x^+).$$

Otherwise

$$\alpha(x^-) < \alpha(x^+).$$

Now, in the equation (1), the closed intervals $[x_{i-1}, x_i]$ are replaced by the open intervals (x_{i-1}, x_i) and thus

$$M_i = \sup\{f(x) : x_{i-1} < x < x_i\}$$

and

$$m_i = \inf\{f(x) : x_{i-1} < x < x_i\}. \tag{6}$$

(2) is defined as

$$\begin{aligned} U(P,f,\alpha) &= \sum_{i=0}^{n} f(x_i)[\alpha(x_i^+) - \alpha(x_i^-)] + \sum_{i=1}^{n} M_i[\alpha(x_i^-) - \alpha(x_{i-1}^+)] \\ L(P,f,\alpha) &= \sum_{i=0}^{n} f(x_i)[\alpha(x_i^+) - \alpha(x_i^-)] + \sum_{i=1}^{n} m_i[\alpha(x_i^-) - \alpha(x_{i-1}^+)]. \end{aligned} \tag{7}$$

The upper and lower Riemann-Stieltjes integrals of f with respect to α, over $[a,b]$ are defined as

$$\int_a^{-b} f\,d\alpha = \inf_P U(P,f,\alpha) \quad \text{and} \quad \int_{-a}^{b} f\,d\alpha = \sup_P L(P,f,\alpha), \tag{8}$$

where the inf and sup being taken over all partitions. We denote their common value by

$$\int_a^b f\,d\alpha \tag{9}$$

if (8) are equal. This is the Ross' definition of the Riemann-Stieltjes integral of f with respect to α, over $[a,b]$.

Theorem 2. [**2**, Theorem 35, 20] *If f is Riemann-Stieltjes integrable on $[a,b]$ with respect to α in the usual sense, then f is Riemann-Stieltjes integrable on $[a,b]$ with respect to α in the Ross' type sence.*

Remark. *Also, Ross* [3] *proposed another definition which is a generalized limit of sums definition, and showed that his two definitions are equivalent.*

3. The Lebesgue Integral

First we recall some definitions. If $A \subset R$ is the union of a finite number of intervals, then A is said to be an *elementary set*. Let $\mathcal{E}$ denote the family of all elementary subsets of R. Note that $\mathcal{E}$ is a ring. For the monotonically increasing function α on R, we define the function μ on bounded intervals by

$$\begin{aligned} \mu([a,b)) &= \alpha(b^-) - \alpha(a^-) \\ \mu([a,b]) &= \alpha(b^+) - \alpha(a^-) \\ \mu((a,b]) &= \alpha(b^+) - \alpha(a^+) \\ \mu((a,b)) &= \alpha(b^-) - \alpha(a^+). \end{aligned} \tag{10}$$

For the elementary set $A = I_1 \cup \cdots \cup I_n$, we also define

$$\mu(A) = \mu(I_1) + \cdots + \mu(I_n) \tag{11}$$

if these intervals are pairwise disjoint. By [5, 11.6(b)] the nonnegative finitely additive set function μ defined by (10) and (11) is *regular*, in the sense that to very $A \in \mathcal{E}$ and to every $\varepsilon > 0$ there exists sets $F \in \mathcal{E}$, $G \in \mathcal{E}$ such that F is closed, G is open, $F \subset A \subset G$ and

$$\mu(G) - \varepsilon \leq \mu(A) \leq \mu(F) + \varepsilon.$$

By [5, Theorem 11.10] μ can be extended to a measure on a σ-algebra. This measure will also be denoted by μ.

The nonnegative function s defined on $\mathbf{R}$ is a *simple function* if the range of s is finite. Suppose $s(\mathbf{R}) = \{c_1, \ldots, c_n\}$. Let $E_i = \{x \in [0,1] : s(x) = c_i\}$ for $i = 1, 2, \ldots, n$. Then every simple function is a finite linear combination of characteristic function. More explicitly, we have

$$s(x) = \sum_{i=1}^{n} c_i K_{E_i}(x), \quad x \in [0,1], \quad c_i > 0, \tag{12}$$

where

$$K_{E_i}(x) = \begin{cases} 1 & \text{if } x \in E_i \\ 0 & \text{if } x \notin E_i, \end{cases} \qquad i = 1, 2, \ldots, n.$$

Assume that $s(x)$ is measurable, i.e. $\{x \in \mathbf{R} : s(x) > a\}$ is measurable for every nonnegative real number a.

Now we define

$$I(s) = \sum_{i=1}^{n} c_i \mu(E_i). \tag{13}$$

If the nonnegative function f defined on $\mathbf{R}$ is measurable, then we define

$$\int_a^b f \, d\mu = \sup I(s), \tag{14}$$

where the sup is taken over all measurable simple function s such that $0 \leq s \leq f$. The left member of (14) is called the *Lebesgue integral* of f with respect to the measure μ, over $[a, b]$. If (14) exists, we say that f is *Lebesgue integrable* with respect to μ and we write $f \in \mathfrak{L}(\mu)$ on $[a, b]$.

Example 3: Let E be the set of all rational numbers in $[a, b]$. If we consider the characteristic function K_E then, whatever the mode of partition of $[a, b]$, every M_i is 1 and every m_i is 0. Thus there is no Riemann integral. But we have $\int_a^b K_E \, d\mu = 0$ since $\mu(E) = 0$.

4. The Consistency Theorem

Lemma 4 (Lebesgue's monotone convergence theorem). *Let $\{f_n\}$ be a sequence of measurable functions defined on a measurable set E with the property*

$$0 \leq f_1(x) \leq f_2(x) \leq \dots, \qquad x \in E.$$

Let f be defined by

$$f_n(x) \longrightarrow f(x), \qquad x \in E$$

as $n \to \infty$. Then

$$\int_E f_n \, d\mu \longrightarrow \int_E f \, d\mu$$

as $n \to \infty$

Theorem 5. *If $f \in \mathfrak{R}$ on $[a,b]$, then $f \in \mathfrak{L}(\mu)$ on $[a,b]$ and*

$$\int_a^b f \, d\mu = \int_a^b f \, d\alpha$$

Proof: See [**5**, Theorem 11.33] or [**1**, p.28].

Now our main consistency theorem follows. Its proof is based upon the standard proof in [5, Theorem 11.33(a)].

Theorem 6. *If $f \in \mathfrak{R}(\alpha)$ on $[a,b]$, then $f \in \mathfrak{L}(\mu)$ on $[a,b]$ and*

$$\int_a^b f \, d\mu = \int_a^b f \, d\alpha.$$

Proof: Suppose that f is bounded on $[a,b]$. By [**5**, Theorem 6.4] there is a sequence of partition

$$P_k = \{a = x_0 < x_1 < \dots < x_n = b\}$$

of $[a,b]$, such that P_{k+1} is a refinement of P_k, such that $x_i - x_{i-1} < 1/k$ for all $i = 1, 2, \dots, n$, and such that

$$\lim_{k\to\infty} L(P_k, f, \alpha) = \int_{-a}^{b} f \, d\alpha \qquad \text{and} \tag{15}$$

$$\lim_{k\to\infty} U(P_k, f, \alpha) = \int_a^{-b} f \, d\alpha.$$

We define two functions U_k and L_k on $[a,b]$:

$$U_k(a) = L_k(a) = f(a) \tag{16}$$

$$U_k(x) = M_i, \quad L_k(x) = m_i \quad \text{for} \quad x_{i-1} < x < x_i$$

and

$$U_k(x_i) = M_i \frac{\alpha(x_i) - \alpha(x_i-)}{\alpha(x_i+) - \alpha(x_i-)} + M_{i+1} \frac{\alpha(x_i+) - \alpha(x_i)}{\alpha(x_i+) - \alpha(x_i-)} \tag{17}$$

$$L_k(x_i) = m_i \frac{\alpha(x_i) - \alpha(x_i-)}{\alpha(x_i+) - \alpha(x_i-)} + m_{i+1} \frac{\alpha(x_i+) - \alpha(x_i)}{\alpha(x_i+) - \alpha(x_i-)}$$

for $\alpha(x_i-) < \alpha(x_i) < \alpha(x_i+)$. Then we have

$$\int_a^b U_k\,d\mu = \sum_{i=1}^n M_i \Delta\alpha_i = U(P_k, f, \alpha) \tag{18}$$
$$\int_a^b L_k\,d\mu = \sum_{i=1}^n m_i \Delta\alpha_i = L(P_k, f, \alpha).$$

Moreover

$$m \le L_k(x) \le L_{k+1}(x) \le f(x) \le U_{k+1}(x) \le U_k(x) \le M \tag{19}$$

for all $k = 1, 2, \ldots$ and all $x \in [a, b]$, where

$$m = \inf\{f(x) : a \le x \le b\} \qquad \text{and}$$
$$M = \sup\{f(x) : a \le x \le b\}.$$

By (19), there exist

$$U(x) = \lim_{k\to\infty} U_k(x) \quad \text{and} \quad L(x) = \lim_{k\to\infty} L_k(x) \tag{20}$$

with the property

$$m \le L(x) \le f(x) \le U(x) \le M, \qquad x \in [a, b]. \tag{21}$$

Since U_k, L_k, U and L are bounded measurable with respect to the measure μ, they are Lebesgue integrable on $[a, b]$. Therefore

$$\int_a^b U\,d\mu = \lim_{k\to\infty} \int_a^b d\mu \quad \text{and} \quad \int_a^b L\,d\mu = \lim_{k\to\infty} \int_a^b L_k\,d\mu \tag{22}$$

by Lebesgue's dominated convergence theorem. From (15), (18) and (22), if follows that

$$\int_a^b U\,d\mu = \int_a^{-b} \quad \text{and} \quad \int_a^b L\,d\mu = \int_{-a}^b f\,d\alpha. \tag{23}$$

Thus

$$f \in \Re(\alpha) \quad \text{on} \quad [a, b] \Leftrightarrow \int_a^b (U - L)\,d\mu = 0. \tag{24}$$
$$\Leftrightarrow U(x) = L(x) \quad \text{a.e. on} \quad [a, b].$$

But by (21) and (24) we have $L(x) = f(x)$ a.e. on $[a, b]$. Therefore $f \in \mathfrak{L}(\mu)$ on $[a, b]$ because $L \in \mathfrak{L}(\mu)$ on $[a, b]$ and the measure μ is complete, i.e., every subset of a set of μ-measure zero has μ-measure zero. Consequently we have $\int_a^b f\,d\mu = \int_a^b f\,d\alpha$. This completes the proof.

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