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The Likelihood Ratio Test for the Equality of Scale Parameters of Several Exponential Distributions Based on Type II Censored Samples

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1. Introduction

In life testing, distributions of the exponential type are often proposed for the appropriate models of the underlying mechanism. During the past years, there have been significant developments in statistical inferences about the location and scale parameters of the exponential distributions. Sukhatme (1936) initially discussed the likilihood ratio(LR) test for equality of parameters of several exponential distributions based on complete samples. Epstein and Tsao(1953) proposed the LR test for the location and the scale parameters of two exponential distributions based on type II censored samples. For testing the equality of both parameters, an iterative procedure is used by Hogg and Tanis(1963). This procedure is an iteration or repetition of the test proposed by Epstein and Tsao(1953). Snigh(1983) developed the LR test for the equality of location parameters of several exponential distributions based on type II censored samples. The main advantages of the LR test lies in its relative simplicity and convenience.

In this paper, we consider the problem of the LR test for the equality of scale parameters of several exponential distributions which have the same location parameters but are suspected of having different scale parameters.

Let the i-th exponential probability density function be given by

$$(1.1)\qquad f(x_i) = \begin{cases} \dfrac{1}{\sigma_i}\exp\left\{-\dfrac{x_i-a}{\sigma_i}\right\}, & x_i \geq a, \quad \sigma_i > 0, \quad -\infty < a < \infty, \\ & i = 1, 2, \ldots, k+1, \\ 0, & \text{otherwise} \end{cases}$$

where σ_i is the scale parameter and a the common location parameter. In life testing, the location parameter a is sometimes thought of as the guarantee period within which no failures can occur while the scale parameter σ_i is the mean life beyond that guarantee period.

A test for the equality of scale parameters of $k+1$ (≥ 2) exponential distribution consists of testing the null hypothesis

$$(1.2)\qquad H_0 : \sigma_1 = \sigma_2 = \cdots = \sigma_{k+1} = \sigma, \quad a \text{ is unspecified}$$

versus the alternative hypothesis

$$(1.3)\qquad H_1 : \text{at least two } \sigma\text{'s are unequal.}$$

For this test, some distributional results are discussed in section 2. In section 3, the LR test for testing H_0 versus H_1 based on type II censored samples is considered and the distribution of the LR test statistic is derived. Section 4 describes concluding remarks.

2. Preliminaries

Some distributional properties of a random sample drawn from a exponential type distribution are discussed in the following lemmas.

Lemma 2.1. *Let $X_{i1}, X_{i2}, \ldots, X_{in_i}$ be an order statistic of size n_i in the i-th population from* (1.1) *and let $X_{i1}, X_{i2}, \ldots, X_{ir_i}$ be the r_ith smallest order statistic. Define*

$$V_i = \frac{2}{\sigma_i}\left\{\sum_{j=1}^{r_i}(X_{i1} - X_{ij}) + (n_i - r_i)(X_{ir_i} - X_{i1})\right\}, \qquad i = 1, 2, \ldots, k+1, \tag{2.1}$$

and

$$W_j = \frac{2n_j}{\sigma_j}(X_{j1} - X_{11}), \qquad X_{j1} > X_{11}, \qquad j = 2, 3, \ldots, k+1. \tag{2.2}$$

Then, V_i's are independently chi-square distributed with degrees of freedom $(2r_i - 2)$ and W_j's are also independently chi-square distributed with degrees of freedom 2. Furthermore, V_i's and W_j's are independent.

Proof: See Epstein and Tsao (1953).

Lemma 2.2. *Let $X_1, X_2, \ldots, X_{k+1}$ be mutually independent random variables, each having a chi-square distribution with degrees of freedom $v_1, v_2, \ldots, v_{k+1}$, respectively. Define*

$$Y_i = \frac{X_i}{\sum_{i=1}^{k+1} X_i}, \qquad i = 1, 2, \ldots, k+1. \tag{2.3}$$

The, the joint distribution of $(Y_1, Y_2, \ldots, Y_k)$ follows a Dirichelet distribution with parameters $\frac{v_1}{2}, \frac{v_2}{2}, \ldots, \frac{v_{k+1}}{2}$.

Proof: See Johnson and Kotz (1976).

Lemma 2.3. *If the joint distribution of $(Y_1, Y_2, \ldots, Y_k)$ is a Dirichlet with parameters $d_1, d_2, \ldots, d_{k+1}$, where $Y_{k+1} = 1 - Y_1 - \cdots - Y_k$, and $s_1, s_2, \ldots, s_{k+1}$ are positive integers, then the probability density function of $\prod_{i=1}^{k+1} Y_i^{s_i}$ is given in terms of an H-function as*

$$\frac{\Gamma\left(\sum_i d_i\right)}{\prod_{i=1}^{k+1}\Gamma(d_i)} H_{1,k+1}^{k+1,0}\left[x \,\middle|\, \begin{matrix} \left(\sum d_i - \sum s_i, \sum s_i\right) \\ (d_1 - s_1, s_1), (d_2 - s_2, s_2), \ldots, (d_{k+1} - s_{k+1}, s_{k+1}) \end{matrix}\right] \tag{2.4}$$

where H-function is defined on Mathai(1971).

Proof: See Rogers and Young (1973).

3. Likelihood Ratio Test

In life testing, experiments are discontinued as soon as a fixed number r_i of items fails : that is, in a random sample of size n_i from the i-th population, we sometime observe only the first r_i order statistic. This random sample is called type II censored samples. Based on this type II censored samples, the LR test for testing H_0 versus H_1 is derived in lemma 3.1 and lemma 3.2

Lemma 3.1. *Let* $H_{in_i} = \{X_{i1} < X_{i2} < \cdots < X_{in_i}, \quad i = 1, 2, \ldots, k+1\}$ *be an ordered random sample from* (1.1) *and let* H_{ir_i} *be the set of* r_i *smallest observation of* H_{in_i}. *Suppose that these samples are ordered such as* $X_{j1} > X_{11}$, $j = 2, 3, \ldots, k+1$. *Then, for testing* H_0 *based on type II censored sample* H_{ir_i}, *the LR test is that we reject* H_0 *in favor of* H_1 *if*

$$\lambda = K \prod_{i=1}^{k+1} Y_i^{r_i} \leq \lambda_0 \tag{3.1}$$

where λ_0 *is some fixed constant satisfying* $0 \leq \lambda_0 \leq 1$, *and*

$$Y_i = \frac{S_i}{\sum_{i=1}^{k+1} S_i}, \quad i = 1, 2, \ldots, k+1, \tag{3.2}$$

$$S_i = \sum_{j=1}^{r_i} (X_{ij} - X_{11}) + (n_i - r_i)(X_{ir_i} - X_{11}), \quad i = 1, 2, \ldots, k+1, \tag{3.3}$$

$$K = \prod_{i=1}^{k+1} \left(\frac{R}{r_i}\right)^{r_i}, \tag{3.4}$$

and

$$R = \sum_{i=1}^{k+1} r_i. \tag{3.5}$$

Proof: The likelihood function, based on type II censored samples, is given by

$$\begin{aligned} &L(\sigma_1, \sigma_2, \ldots, \sigma_{k+1}, a \mid X_{11}, X_{12}, \ldots, X_{1r_1}; \ldots; X_{k+11}, \ldots, X_{k+1r_{k+1}}) \\ &= \prod_{i=1}^{k+1} \frac{n_i!}{(n_i - r_i)!\, \sigma_i^{r_i}} \exp\left[-\frac{1}{\sigma_i}\left\{\sum_{j=1}^{r_i} (X_{ij} - a) + (n_i - r_i)(X_{ir_i} - a)\right\}\right] \end{aligned} \tag{3.6}$$

In the parameter space $\Omega = \{(\sigma_1, \sigma_2, \ldots, \sigma_{k+1}, a)\,;\, -\infty < a < \infty, \sigma_i > 0, i = 1, 2, \ldots, k+1\}$, the maximum likelihood(ML) estimator of a and σ_i are obtained as follows.

$$\hat{a} = X_{11} \tag{3.7}$$

and

$$\hat{\sigma}_i = \frac{S_i}{r_i}, \qquad i = 1, 2, \ldots, k+1. \tag{3.8}$$

Similarly, the ML estimator of a and σ in the parameter space $w = \{(\sigma_1, \sigma_2, \dots, \sigma_{k+1}, a)\,;\ -\infty < a < \infty, \sigma_1 = \sigma_2 = \cdots = \sigma_{k+1} = \sigma > 0\}$ are

$$\hat{a} = X_{11}$$

and

$$\hat{\sigma} = \frac{1}{R}\sum_{i=1}^{k+1} S_i. \tag{3.9}$$

Therefore, the likelihood ratio, denoted by λ, is given by $\lambda = K\sum_{i=1}^{k+1} Y_i^{r_i}$ and hence the proof is complete.

This LR test is equivalent to test that reject H_0 in favor of H_1 if $\prod_{i=1}^{k+1} Y_i^{r_i} \leq c$ where c is a constant. It remains to determine c so that the critical region has the desired size of test. The distribution of the LR test statistic is needed to find the exact value of c.

Lemma 3.2. *Suppose that all the conditions in lemma 3.1 hold. Then, the propbability density function of the likelihood ratio test statistic $\prod_{i=1}^{k+1} Y_i^{r_i}$ is given in terms of an H-function as*

$$\Lambda = \frac{\Gamma(R-1)}{\Gamma(r_1-1)\Gamma(r_2)\dots\Gamma(r_{k+1})} H^{k+1,0}_{1,k+1}\left[x \,\Big|\, \begin{matrix}(-1,R)\\ (1,r_1),(0,r_2),\dots,(0,r_{k+1})\end{matrix}\right] \tag{3.10}$$

where H-function is defined on Mathai(1971).

Proof: Define

$$T_i = \sum_{j=1}^{r_i}(X_{ij} - X_{i1}) + (n_i - r_i)(X_{ir_i} - X_{i1}), \qquad i = 1, 2, \dots, k+1. \tag{3.11}$$

Then, by lemma 2.1, $\frac{2}{\sigma_i}T_i$'s are independently chi-square distributed with degrees of freedom $2(r_i - 1)$. Furthermore, from the definition of W_j in (2.2), S_i in (3.3), and T_i in (3.11), S_i's can be expressed by

$$S_1 = T_1$$

and

$$S_j = T_j + \frac{\sigma_j}{2}W_j, \qquad j = 2, 3, \dots, k+1. \tag{3.12}$$

Therefore, by lemma 2.1, $\frac{2}{\sigma_1}S_1$ and $\frac{2}{\sigma_j}S_j$, $j = 2, 3, \dots, k+1$, are independently chi-square distributed with degrees of freedom $2(r-1)$ and $2r_i$, respectively. Provided $\sigma_1 = \sigma_2 = \cdots = \sigma_{k+1} = \sigma$, the sum of $\frac{2}{\sigma}S_i$, $\frac{2}{\sigma}\sum_{i=1}^{k+1} S_i$, is then chi-square distributed with degree of freedom $2(R-1)$ and hence, by lemma 2.2, the joint distribution of $(Y_1, Y_2, \dots, Y_k)$ is to have a Dirichlet distribution with parameters $r_1 - 1, r_2, \dots, r_k, r_{k+1}$. The proof immediately follows from lemma 2.3.

4. Concluding Remarks

This paper gives the LR test for the equality of scale parameters of several exponential distributions based on type II censored samples and shows that the LR test is to reject H_0 in favor of H_1 at α level of significance if $\prod_{i=1}^{k+1} Y_i^{r_i} \leq c$, where a constant c is determined so that $\Lambda \geq \Lambda_\alpha$, where Λ_α is the upper 100 percentile of Λ function. This LR test reduces to that of Epstein and Tsao(1953) if we compare two scale parameters σ_1 and σ_2. Since the iterative procedure of Hogg and Tanis(1963) involves the sequence of nested alternative hypothesis, it is very tedious and inconvinient. Unlikely the interated procedure, the LR test is straightforward and sometime optimal because of the desirable features of the LR test.

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