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Note on a Semigroup with Idempotent Principal Ideals

Young-Hee Kim

ChungBuk National University, Cheong Ju, Korea

1. Introduction

The purpose of this paper is to investigate the structure of some special idempotent semigroups, called a semigroup with idempotent principal ideals, which would be considered to form a category of idempotent semigroups.

A global structure theorem on idempotent semigroup was carried out by David Mclean [1]. Before going into the theorem, we need the following definitions. A semigroup S is called an idempotent semigroup if every element of S is idempotent. Define $S(a) = \{x \in S | axa = a \text{ and } xax = x\}$, where $a \in S$. An idempotent semigroup S is called (1) left regular, (2) right regular, (3) regular, if it satisfies the following corresponding identities:

(1) $$aba = ab,$$

(2) $$aba = ba,$$

(3) $$abaca = abca$$

Theorem 1.1 [5]. *Let S be an idempotent semigroup. Then we have the following:*

(1) *$S(a)$ is a semigroup, for any $a \in S$.*
(2) *If $x \in S(a)$, then $S(a) = S(x)$.*
(3) *Let $a, b \in S$. Then $S(a)S(b)S(ba) = S(ab)$.*
(4) *$S(a) = S(a)aS(a)$ and $(xay)(uav) = xav$ for $xay, uav \in S(a)aS(a)$.*

Theorem 1.2[7]. *Let S be an idempotent semigroup and let $a \in S$. Let $S(a) = \{x_i : i \in I\}$. Then we have the following.*

(1) *If $x_i a x_j = x_i$, then $x_j a x_i = x_j$.*
(2) *If $x_i a x_j = x_j$, then $x_j a x_i = x_i$.*
(3) *If $x_i x_j = x_k$, then $x_k x_i = x_i, x_j x_k = x_j, x_i x_k = x_k$ and $x_k x_j = x_k$.*

Definition 1.3: A semigroup with idempotent ideals is a semigroup in which every ideal is idempotent.

Definition 1.4: If a is an element of a semigroup S, the smallest left (right) ideal containing a is $SaU\{a\}$, which we may conveniently write as $S^1a(aS^1)$, and which we shall call the principal left (right) ideal generated by a. A principal ideam (a) of S means if it satisfies S^1a and aS^1.

2. Main Results

Theorem 2.1. *The principal ideal (a) of a semogroup S is idempotent if and only if $a \in SaSaS$.*

Proof: Assume $a \in SaSaS$. Then

$$a \in (Sa)(SaS) \subseteq (a)(a) = (a)^2,$$

whence $(a) \subseteq (a)^2$. As the converse inclusion is always true, $(a)(a)^2$.

Assume now $(a) = (a)^2$. Then

$$\begin{aligned} a \in (a) = (a)^5 &= S^1aS^1 \cdot S^1aS^1 \cdot S^1aS^1 \cdot S^1aS^1 \cdot S^1aS^1 \\ &= S^1aS^1S^1 \cdot a \cdot S^1S^1aS^1S^1 \cdot a \cdot S^1S^1aS^1 \\ &\subseteq SaSaS, \end{aligned}$$

as asserted.

Theorem 2.2. *Let S be a semigroup and I an ideal of S. If $(a) = (a)^2$ for every element of I, then $I = I^2$.*

Proof: Assume $(a) = (a)^2$ for each $a \in I$. Then $a \in (a)^2 \subseteq I^2$ for each $a \in I$ Hence $I \subseteq I^2$. on the other hand, $I^2 \subseteq I$ for every ideal I.

The statement converse to this theorem does not hold. This assertion can be verified by the following example.

Example 1: Let $S = \{0, 1, a\}$ be a commutative semigroup in which $a^2 = 0$. Consider an ideal $I = S$. Then $I^2 = I$, but $(a)^2 = (0) \neq (a)$ for each element a of an ideal I.

Corollary 2.3. *A commutative semigroup is regular if and only if it is a semigroup with idempotent ideals.*

Proof: If S is a semigroup with idempotent ideals, then every element a of S can be represented in the form

$$a = xayaz = a(xyz)a \qquad (x, y, z \in S)$$

by Theorem 2.1 and by commutatity. Thus, S is regular.

conversely, assume that S is regular and let a be an arbitrary element of S. Then there exists an $x \in S$ such that $axa = a$ and

$$a = (ax)a \in (a)(a) = (a)^2$$

Hence $(a) = (a)^2$ and thus S is a semigroup with idempotent ideals by Theorem 2.2.

Corollary 2.4. *Every ideal of a regular semigoup is idempotent.*

Proof: We verified in the second half of the proof of corollary 2.3.

Theorem 2.5. *The following assertions are equivalent:*

(1) *S is a semigroup with idempotent ideals;*
(2) *S is a semigroup with idempotent principal ideals;*
(3) *a $SaSaS$ for every element a of the semigroup S.*

Proof: (1) implies (2) trivially. (2) implies (3) by Theorem 2.1. (3) implies (1): if (3) is satisfied, then every principal ideal of S is idempotent by Theorem 2.1 and thus every ideal of S is idempotent, by Theorem 2.2

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