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Cumulative Weighted Score Control Schemes for Controlling the Mean of a Continuous Production Process

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ABSTRACT

Cumulative sum schemes based on a weighted score are considered for controlling the mean of a continuous production process; in which both the one-sided and two-sided schemes are proposed. The average run lengths and the run length distributions for the proposed schemes are obtained by the Markov chain approach. Comparisons by the average run length show that the proposed schemes perform nearly as well as the standard cumulative sum schemes in detecting changes in the process mean. Comparisons of the one-sided schemes by the run length distribution are also presented.

1. Introduction

This paper considers the problem of detecting changes in the mean of a continuous Production Process. Samples of fixed size n are taken successively form the process at regular time intervals. The proposed schemes use the sample means $\overline{X}_j$, $j=1,2,\cdots$, which are independent and normally distributed with mean μ and finite known variance σ^2/n. In this paper, without loss of generality, we will assume that $\mu_0=0$ and $\sigma^2/n=1$, where μ_0 is the *target value* of the process mean.

The cumulative sum(CUSUM) schemes originally proposed by Page(1954) have been widely used in industrial situations due to their sensitivity in detecting deviations in the

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process mean from the *target value* μ_0. The procedure for *one-sided* CUSUM schemes to detect increases in the process mean, called *high side* CUSUM schemes, is based on the CUSUM's

$$S_H(j)=\max[0,\ S_H(j-1)+\overline{X}_j-K],\quad j=1,2,\cdots\cdots, \tag{1.1}$$

where $S_H(0)=0$ and corrective action is taken when $S_H(j)\geq h$, where h is the so-called *decision interval* of the procedure. Also the positive real number K is the *reference value* of the procedure, The reference value K is usually set equal to $\delta/2$ since this value minimizes the average run length(ARL) at a mean shift of δ from the target value for a given ARL at the target value(see Bissell, 1959; Ewan and Kemp, 1960; Reynolds, 1975). The procedure for *low side* CUSUM schemes to detect decreases in the process mean is based on the CUSUM's

$$S_L(j)=\min[0,\ S_L(j-1)+\overline{X}_j+K],\quad j=1,2,\cdots, \tag{1.2}$$

where $S_L(0)=0$ and corrective action is taken when $S_L(j)\leq -h$.

To detect changes in the process mean in either direction, two separate one-sided schemes based on the CUSUM's in(1.1) and (1.2) which are run concurrently are customarily used. This *two-sided* CUSUM scheme signals for corrective action either when $S_H(j)\geq h$ or when $S_L(j)\leq -h$.

Munford(1980) proposed cumulative score control schemes in which scores of $+1, -1$ or 0 assigned to sample means are cumulated. Ncube and woodall(1984) proposed combined Shewhart-cumlative score control schemes by adding the Shewhart action limits to the Munford's schemes. Despite of the simplicity in calculating the ARL's for these schemes, they are much less sensitive to shifts in the process mean than the standard CUSUM schemes. To increase the sensitivity we consider two procedures in which a weighted score $w(\geq 2)$ is used to augment the Ncube and Woodall procedure. Like the Ncube and Woodall's procedures, each of our procedures can signal an out-of-control situation at any stage.

The one-sided cumulative score schemes based on the weighted score are proposed in Section 2. Comparisons by the ARL with the standard CUSUM schemes are made. Comparisons by the run length distribution are also presented. In section 3, the two-sided schemes are developed and compared with the standard CUSUM schemes in terms of the ARL. The ARL's and the run length distributions for the proposed schemes are exactly obtained by the Markov chain approach.

2. One-Sided Control Schemes

2. 1 Control Procedures

In this section we are concerned with the problem of detecting changes in the process mean in the positive direction. If we want to detect changes in the mean in the negative direction, the low side schemes can be obtained by slightly altering the high side schemes for detecting increases in the mean.

To costruct new procedures for the one-sided control schemes, we classify the sample mean $\overline{X}_i$ into an integer-valued random variable Y_i as follows:

$$Y_i = \begin{cases} 2h, & k_3 < \overline{X}_i - k, \\ w, & k_2 < \overline{X}_i - k \leq k_3, \\ 1, & k_1 < \overline{X}_i - k \leq k_2, \\ 0, & k_1 \leq \overline{X}_i - k \leq - k_1, \\ -1, & -k_2 \leq \overline{X}_i - k < k_1, \\ -w, & \overline{X}_i - k < -k_2, \end{cases} \qquad i=1,2,\cdots, \tag{2.1}$$

where K, k_1,k_2 and k_3, ($k_1<k_2<k_3$, $3<k_3$) are positive real numbers, and w and $h(2\leq w<h)$ are positive integers. Note that K and h are reference value and the decision interval for the control procedures, respectively.

We classify the sample means into the scores in (2.1) for three reasons. First, it seems reasonable to assign a score -1, 0 or 1 to each sample mean according as it falls between about -1.5 and -0.5, -0.5 and 0.5, 0.5 and 1.5, respectively. Second, it is desirable to assin a heavy score($\geq$2or$\leq$-2) according as it falls between about 1.5 and k_3 or falls below about -1.5. Third, it is also desirable to assign a score $2h$ as it falls outside k_3 so that a rectifying action can be quickly indicated when the process mean has shifted.

Under the assumptions of the distributions of $\overline{X}_i$, $i=1, 2, \cdots$, the random variables Y_i, $i=1,2,\cdots$, are mutually independent and identically distributed with probability mass function

$$
\begin{aligned}
&\Pr(Y_i=2h)= 1-\phi(K+k_3-\mu)= p_6,\\
&\Pr(Y_i=w)= \phi(K+k_3-\mu)-\phi(K+k_2-\mu)= p_5,\\
&\Pr(Y_i=1)= \phi(K+k_2-\mu)-\phi(K+K_1-\mu)= p_4,\\
&\Pr(Y_i=0)=1-(p_1+p_2+p_4+p_5+p_6)=p_3,\\
&\Pr(Y_i=-1)= \phi(K-k_1-\mu)-\phi(K-k_2-\mu)=p_2\\
&\Pr(Y_i=-w)= \phi(K-k_2-\mu)=p_1 \quad ,\\
&\qquad i=1,2,\cdots
\end{aligned}
\tag{2.2}
$$

where $\phi(\cdot)$ is the standard normal cumulative distribution function.

We consider the following two types of control procedures for the one-sided schemes.

Rule I: $S_j=\max[0, S_{j-1}+Y_j], j=1,2,\cdots,$

and

Rule II: $S_j=\begin{cases} 0, & S_{j-1}+Y_j<-h \\ S_{j-1}+Y_j, & \text{otherwise,} \end{cases}$

$j=1,2,\cdots,$

where in either case, $S_0=0$ and corrective action is taken when $S_j\geq h$. Our procedures without the weighted score, w, are the combined-Shewhart cumulative score control procedures of Ncube and Woodall(1984). If $k=0$ and $k_2=\infty$, our procedures become the cumulative score control procedures of Munford(1980). If $h=1$, in addition, they correspond to the Shewhart(1931) control procedure.

2.2. Markov Chain Approach

In order to obtain the ARL's for our schemes, we apply the Markov chain aproach. It is easily seen that both the sequences of our random variables S_j, $j=1,2,\cdots$, in Rule I and Rule II form Markov chains. The state spaces of these Markov chains are $\{0, 1, \cdots, h\}$ for Rule I and $\{-h+1, -h+2,\cdots\cdots,h\}$ for Rule II, where the decision interval h is the *absorbing state*. In this paper, our interest is restricted to the case where the cumulative scores(CuScore's) start out at state zero.

The explicit form of the transition probability matrix P for Rule II, as a example, for the case where $w=2$ and $h=4$ is given by

$$
P=\begin{array}{c} \\ -3 \\ -2 \\ -1 \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array}
\begin{array}{c}
\begin{array}{ccccccc|c} -3 & -2 & -1 & 0 & 1 & 2 & 3 & 4 \end{array} \\
\left[\begin{array}{ccccccc|c}
p_3 & p_4 & p_5 & p_1+p_2 & 0 & 0 & 0 & p_6 \\
p_2 & p_3 & p_4 & p_1+p_5 & 0 & 0 & 0 & p_6 \\
p_1 & p_2 & p_3 & p_4 & p_5 & 0 & 0 & p_6 \\
0 & p_1 & p_2 & p_3 & p_4 & p_5 & 0 & p_6 \\
0 & 0 & p_1 & p_2 & p_3 & p_4 & p_5 & p_6 \\
0 & 0 & 0 & p_1 & p_2 & p_3 & p_4 & p_5+p_6 \\
0 & 0 & 0 & 0 & p_1 & p_2 & p_3 & p_4+p_5+p_6 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{array}\right]
\end{array}
$$

If we represent the transition probability matrix P for Rule I and Rule II in the partitioned form, P becomes

$$
P=\left[\begin{array}{c|c} R & \underset{\sim}{v} \\ \hline \underset{\sim}{0}^T & 1 \end{array}\right], \tag{2.3}
$$

where $\underset{\sim}{v}$ is a vector of h elements for Rule I and of $2h-1$ elements for Rule II. Also the vector $\underset{\sim}{0}^T$ is the transpose of a vector which has each of its elements equal to zero. R is an $h\times h$ matrix for Rule I and a $(2h-)\times(2h-1)$ matrix for Rule II.

Let N_i be the run lenth for the scheme which starts out at state i and let $L_i=E(N_i)$, the expected value of N_j. From Cox and Miller(1965), then, it is easily seen that the ARL's for the proposed schemes can be obtained from the vector.

$$
\underset{\sim}{L}=(I-R)^{-1}\underset{\sim}{1}, \tag{2.4}
$$

and the run length distributions can also be obtained from the following matrix operations

$$
\underset{\sim}{N}_1=(I-R)\underset{\sim}{1} \tag{2.5}
$$

and

$$
\underset{\sim}{N}_r=R\underset{\sim}{N}_{r-1}=R^{r-1}\underset{\sim}{N}_1,\ r=2,3,\cdots\cdots \tag{2.6}
$$

As an example, the vector N_r for the Rule I schemes is represented by

$$
N_r=\{P_r(N_0=r),\ P_r(N_1=r),\ \cdots\cdots,\ P_r(N_{h-1}=r)\}^T.
$$

2.3. Choice of Schemes

Now we want to choose the appropriate values of the weighted score w, the decision interval h and the classifying real numbers k_1, k_2 and k_3, so that the proposed schemes will have desirable properties. The schemes should be chosed to give low ARL's at unsatisfactory quality levels for given in-control ARL's , so that changes in the process mean could be quickly detected. In this section, we are concerned with the case where the schemes give minimum ARL(0.5) values for given approximate ARL(0) values of 100, 590 and 940. Note that ARL(0.5) is the ARL when the process mean remains at the value 0.5, the amount of shift in the process mean. In order to find these schemes, we use a reference value K=0.25 since this value seems to minimize the ARL when $\mu=0.5$ for a given ARL when $\mu=0$.

For given w=2,3,4 and 5, first, we find numerically the values of k_1, k_2, k_3 , and h in the order named at in-control situation which give the ARL(0) values of

$$\begin{aligned} 100.0 &\leq \text{ARL}(0) < 100.2, \\ 590.0 &\leq \text{ARL}(0) < 590.4, \\ 940.0 &\leq \text{ARL}(0) < 941.0 \end{aligned} \tag{2.7}$$

with fine grids. In each condition in (2.7), the relative error of the actual ARL(0) value from the nominal ARL(0) value will be less than 0.002. The fine grids are $k_1=0.4(0.01)1.0$, $k_2=1.4(0.01)2.0$, $k_3=3(0.1)5$ and $h=3(1)20$, where a(b)c denotes a, a+b, a+2b, $\cdots$, c.

Once the values of k_1, k_2, k_3, and h for given w under each condition in (2.7) are determined, we can evaluate the ARL(0.5) values using the determined values.

Under each condition in (2.7), finally, we choose the values of w, K_1, k_2, k_3 and h of which the scheme gives the lowest ARL(0.5) value. If a scheme has this property, it will be called the optimum scheme for a given ARL(0) value.

2.4. ARL Comparisons

We compare three types of schemes for approximate ARL(0) values of (a) 100, (b) 590 and (c) 940. First, comparisons for the case where the schemes initially start out at the target value, zero, are given. In Table 1, we list the ARL's for the CUSUM, the optimum Rule I and the optimum Rule II schemes together with their design points. The

ARL's for the CUSUM schemes are obtained from the Markov chain approximation by an extrapolation technique, which is presented in Appendix.

Table 1. ARL Comparisons of One-Sided Control Schemes

(a) ARL(0)=100

Scheme	Design	Point	ARL(0.0)	ARL(0.5)	ARL(1.0)	ARL(2.0)
Rule I:	h=5 K=0.25 h=2	k_1=0.49 k_2=1.50 k_3=3.2	100	15.56	6.97	3.37
Rule II:	h=4 K=0.25 w=2	k_1=0.45 k_2=1.50 k_3=3.4	100	14.59	5.87	2.80
CUSUM:	h=4.42	K=0.25	100	14.85	6.62	3.17

(b) ARL(0)=590

Scheme	Design	Point	ARL(0.0)	ARL(0.5)	ARL(1.0)	ARL(2.0)
Rule I:	h=8 K=0.25 w=2	k_1=0.61 k_2=1.52 k_3=3.4	590	29.43	11.68	5.18
Rule II:	h=6 K=0.25 h=2	k_1=0.60 k_2=1.59 k_3=4.5	590	26.51	9.44	4.44
CUSUM:	h=7.58	K=0.25	590	27.10	10.83	4.97

(c) ARL(0)=940

Scheme	Design	Point	ARL(0.0)	ARL(0.5)	ARL(1.0)	ARL(2.0)
Rule I:	h=9 K=0.25 h=2	k_1=0.52 k_2=1.50 k_3=4.2	940	32.83	13.01	6.31
Rule II:	h=7 K=0.25 w=2	k_1=0.55 k_2=1.52 k_3=3.9	940	29.74	10.47	4.92
CUSUM:	h=8.47	K=0.25	942	30.63	12.02	5.48

From Table 1, without rerard to the ARL(0) values considered, we can see the following facts: (i) The Rule I schemes perform nearly as well as the CUSUM schemes. They also have uniformly lower ARL's than the Munford's(1980) and the Ncube and Woodall's (1984) schemes except the case ARL(0)=100 and μ =2.0 (compare our Table with Table 1 of Ncube and Woodall, 1984), (ii) The Rule II schemes appear as if they are even more sensitive than the CUSUM schemes.

Now we check the performance of the Rule II schemes by comparing the so-called *steady state* ARL's (Crosier, 1986). A shift in the process mean in the industrial situation frequently occurs after the process has been operating for some time. The steady state ARL is the average additional run length after the mean shift occurs. All the CUSUM and CuScore's of the compared schemes may not be zero when the shift occurs. Especially, the CuScore of Rule II probably tends to be on the low side when the process is in-control, so the rule II schemes may take longer to detect than the other two types of schemes. So we need to compare the steady state ARL's. Table 2 gives the steady state ARL's obtained by the aerage of 1000 simulations for each case. In this simulation, 32 observations were taken before the mean changed and the sequence of observations was discarded if a false signal occurred during the first 32 observatons.

Table 2. Steady State ARL Comparisons for ARL(0)=100 by Simulation

Scheme	ARL(0.5)	ARL(1.0)	ARL(2.0)
Rule I:	13.78	5.63	2.86
	(15.56)	(6.97)	(3.37)
Rule II:	14.57	6.28	3.18
	(14.59)	(5.87)	(2.80)
CUSUM	13.09	5.59	2.67
	(14.85)	(6.62)	(3.17)

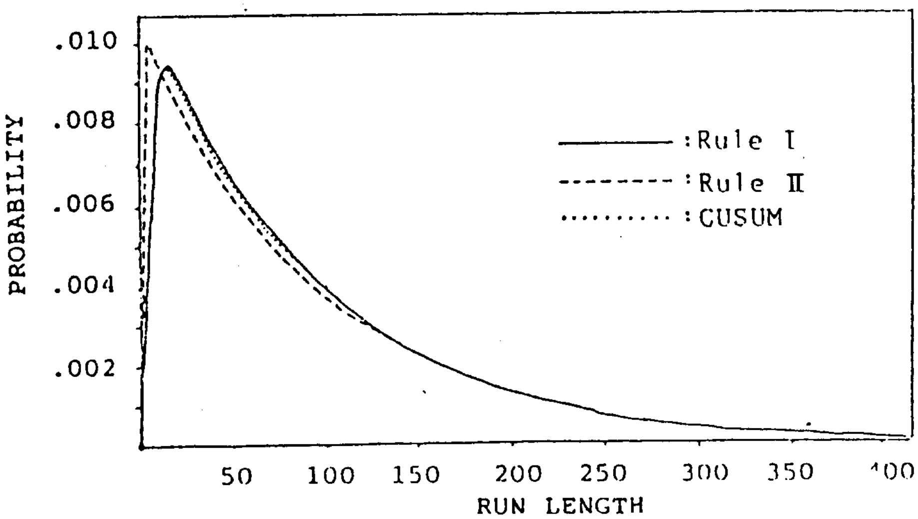
Numbers in parentheses are the ARL's in TABLE 1.

From Table 1 and Table 2, we can conclude that the Rule I schemes performs nearly as well as the CUSUM schemes but the Rule II schemes don't perform so well as the others. The performance of the Rule II schemes in Table 1 appeared better than the others because the ARL's were determined using a starting value of zero. We can expect similar results for ARL(0)=590 and 940.

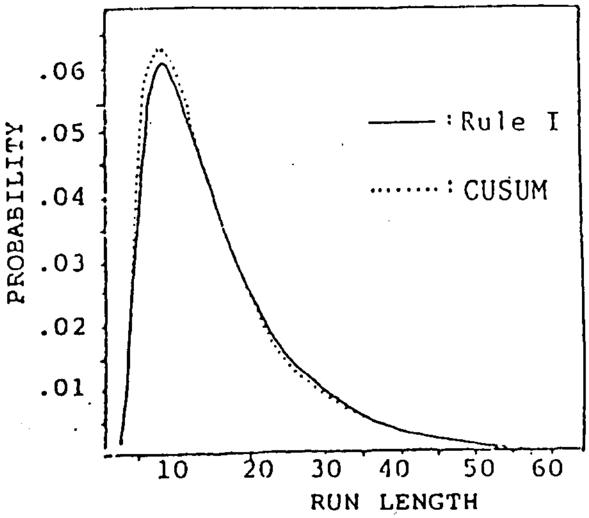
2.5. Comparisons in Terms of Run Length Distribution

Figure 1 shows the plots of the run length distributions for the schemes of three types compared in Table 1. We show the plots for the cases where ARL(0)=100, μ=0.0, 0.5 and 1.0. The run length distributions for the CUSUM schemes were obtained with the Markov chain approximation having 41 states labeled 0,1$\cdots$, 40, where the last state 40 is the absorbing state(see Appendix for details).

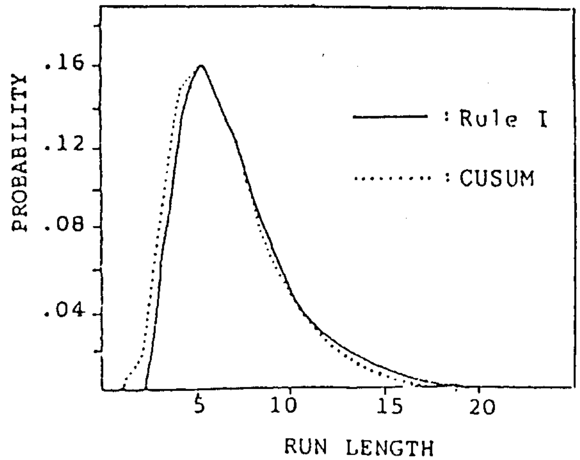
Figure 1. Run Length Distributions for One-Sided Schemes, ARL(0)= ioo



(1) μ =0.0



(2) μ =0.5



(3) $\mu = 1.0$

From Figure 1, we can easily see the following facts: (i) In all cases, the behabiours of the run length distributions for the CUSUM and the Rule I schemes are quite alike. (ii) The Rule II schemes have greater probabilities of short run lengths than the CUSUM and the Rule I schemes in the in-control situation. Note that in the cases where ARL(0)=5 90 and 940, similar results have been derived. Details for other distributions are available from authors.

3. Two-Sided Control Schemes

We now consider schemes in which deviations in the process mean from the target value in either direction call for corrective action to be taken. A two-sided procedure we consider, which will be referred to Rule III, is made up by the two one-sided Rule I's: one to detect increases in the process mean and the other to detect decreases in the process mean. Also, the two one-sided Rule's are run concurrently and separately.

In order to obtain the ARL's for the Rule III schemes, we also apply the Markov chain approach. We can find that two dimensional CuScore's (SH_j, SL_j), $j=1,2,\cdots$, in Rule III, form a Markov chain, where SH_j's and SL_j's are the corresponding Cuscore's to detect increases and decreases in the process mean, respectively. The state space of this Markov chain is $\{(i,j);\ i= 0,1,\cdots, h,\ j=0, -1, \cdots, -h$. A state (i,j) represents SH_j in state i and SL_j in state j. Any state(i,j) with $i=h$ or $j=-h$ is an absorbing state. If we combine all absorbing states into a single absorbing state, the transition probability matrix can be represented by the same form in (2.3). Therefore, the ARL's for the Rule III schemes can also be obtained by the matrix operatin in (2,4).

We want to find the desired schemes when the ARL(0) values are approximately 10 0 and 470. The procedure for finding the optimum schemes is the same as that in Section 2. First, the optimum design points for the high side Rule III schemes are found under the conditions

$$200.0 \leq ARL(0) < 200.3,$$
$$940.0 \leq ARL(0) < 941.0,$$

for the sake of using the results obtained in Section 2. Note that the reference values are also fixed to 0.25. And then, the ARL's for the proposed two-sided schemes are evaluated at these desin points through the matrix operation in (2.4).

In Table 3, we list the ARL's for the two-sided schemes with their design points. The ARL's for the CUSUM schemes are obtained from the Markov chain approximation by extrapolation(see Appendix for details).

Table 3. ARL Comparisons of Two-Sided Control Schemes

(a) ARL(0.0)=100

Scheme	Design	Point	ARL(0.0)	ARL(0.5)	ARL(1.0)	ARL(2.0)
Rule III:	h=6 K=0.25 w=2	k_1=0.53 k_2=1.55 k_3=3.9	104	20.48	8.74	4.26
CUSUM:	K=0.25	h=5.60	104	19.36	8.20	3.84

(b) ARL(0.0)470

Scheme	Design	Point	ARL(0.0)	ARL(0.5)	ARL(1.0)	ARL(2.0)
Rule III:	h=9 K=0.25 w=2	k_1=0.52 k_2=1.54 k_3=4.2	479	32.83	13.01	6.31
CUSUM:	K=0.25	h=8.47	479	30.63	12.03	5.48

From Table 3, without reard to the ARL(0) values considered, we can see the following results: (i) The Rule III schemes have sensitivities close to the CUSUM schemes as in the one-sided case. (ii) From our Table 3 (b) and Table 2 of Ncube and Woodall(1984), it is seen that the proposed two-sided schemes are more sensitive than the Shewhart(1931), the Munford's (1980) and the Ncube and Woodall's schemes.

4. Conclusions

A cumulative weighted score procedure has been proposed and compared to other procedures on the basis of the ARL and the run lenth distribution. The Rule I and the Rule III schemes are found to perform nearly as well as the standard CUSUM schemes. The proposed schemes may be more practical than the CUSUM schemes becase they are simple in dealing with data, that is, it is sufficient to cumulate integers 0, +1, -1, +2 and -2 only after measuring with the eye in which range the observation falls (see Section 2.1). Another advantage of the proposed schemes is the fact that the ARL's and the run length distributions can be exactly obtained, while those of the CUSUM schemes are obtained by Markov chain or Wald's appoximation. Also, the proposed schemes can be applied to a scale problem which would be rather complicated.

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Appendix. Markov Chain Approximation

The use of Markov chains to calculate the approximate ARL's and the run length distributions for the one-sided CUSUM schemes was discussed by Brook and Evans(1972). They replaced the continuous CUSUM scheme by a discrete one that has t+1 possible states, 0, 1, ⋯, t, where t is the absorbing state. The ARL vector for the one-sided CUSUM schemes can be calculated by the matrix operation in (2.4).

In this paper, the ARL's for the one−sided CUSUM schemes were calculated for five different size Markov chains with t values of 10, 20, 30, 40 and 50. For the two-sided CUSUM schemes the sizes of Markov chains are t^2+1, where t=7, 8, 9, 10, 13 and 16. And then, extrapolations to the asymptotic ARL's for $t=\infty$ were accomplished by fitting the following regression formula to the ARL's which were obtained from Markov chains with the stepwise regression search procedure.

$$ARL(t)=ARL(\infty)+A/t+B/t^2+C/t^3+D/t^4. \qquad (A.1)$$

Note that the coefficients of determination after fitting the regression model (A.1) are all greater than 0.99.