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A Lower Confidence Bound on the Probability of a Correct Selection of the t Best Populations[+]

Gyu-Jin Jeong*, Woo-Chul Kim** and Jong-Woo Jeon**

ABSTRACT

When we select the t best out of k populations in the indifference zone formulation, a lower confidence bound on the probability of a correct selection is derived for families with monotone likelihood ratio. The result is applied to the normal means problem when the variance is common, and to the normal variances problem. Tables to implement the confidence bound for the normal variances problem are provided.

1. Introduction

Consider independent observations X_{ij} from each of k populations with cdf's $G(x-\theta_i)$ $(i=1,\cdots,k,\ j=1,\cdots,n)$. When an experimenter wishes to select the t best populations associated wth the t largest parameters, he or she chooses a statstic $Y_i = Y(X_{i1},\cdots,X_{in})$ with cdf $F(y-\theta_i)$ and uses the natural selection rule which selects the populations corresponding to the t largest observations $Y_{(k)},\cdots,Y_{(k-t+1)}$ where $Y_{(1)} \leq \cdots \leq Y_{(k)}$ are the ordered Y_i's $(i=1,2,\cdots,k)$.

For this selection problem, Bechhofer's (1954) indifference zone approach suggests to determine the sample size n, prior to the experiment, to contorl the probability of a correct selection (PCS)

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* Department of Applied Statistics, Hannam University, Daejeon, 301 – 060.

** Department of Computer Science and Statistics, Seoul National Univ., Seoul, 151 – 742.

$$\text{PCS}= \int_{-\infty}^{\infty} \prod_{i=1}^{k-t} F(y-\theta_{[i]})\, d[1-\prod_{j=k-t+1}^{k} \bar{F}(y-\theta_{[j]}) \tag{1.1}$$

where $\theta_{[1]} \leqq \cdots \leqq \theta_{[k]}$ are the ordered parameters and $\bar{F}= 1-F$. As pointed out by Kim (1986), this indifference zone approach is clearly formulated from the point of view of designing an experiment.

Recently, Anderson, Bishop and Dudewicz (1977), Faltin (1980), Kim (1986) and Gupta and Liang (1988) considered a lower confidence bound on PCS when $t=1$ as a method of retrospective analyses. Kim (1986) showed that his lower confidence bound improves upon or generalizes the otheres when $t=1$.

This article generalizes Kim's (1986) result to the case of selecting t best populations. Section 2 presents a lower confidence bound on PCS when the pdf has the monotone likelihood ratio (MLR) property. This result is applied to the normal means problem with common variance in Section 3 and applied to the normal variances problem in Section 4 Tables to implement the lower confidence bound for the normal variances problem are given.

2. A Lower Confidence Bound on PCS

The PCS in (1.1) is easily shown to be non-increasing in $\theta_{[1]},\cdots,\theta_{[k-t]}$ and non-decreasing in $\theta_{[k-t+1]},\cdots,\theta_{[k]}$. Thus we have the inequality

$$\text{PCS} \geqq t \int_{-\infty}^{\infty} F^{k-t}(y+\theta_{[k-t+1]}-\theta_{[k-t]})\, \bar{F}^{t-1}(y)\, f(y)dy \tag{2.1}$$

where f is the pdf of F. It follows from the inequality (2.1) that a *conservative* lower confidence bound on PCS can be obtained by constructing a lower confidence bound of $\theta_{[k-t+1]}-\theta_{[k-t]}$. Such a technique has been used in Anderson, Bishop and Dedewicz (1977) and Kim (1986).

The following lemma is an extension of a result in Kim (1986).

Lemma 1. Assume that $f(y-\theta)$ has the MLR property in y and θ. Then for any fixed $c > 0$, $P_{\underline{\theta}}[Y_{(k-t+1)}- Y_{(k-t)} > c]$ is non-increasing in $\theta_{[1]}$ and non-decreasing in $\theta_{[k]}$.

Proof. By symmetry we may assume $\theta_1 \leqq \cdots \leqq \theta_k$. Let $W_i=Y_{i+1}$, $\eta_i= \theta_{i+1}$ for $i=1,\cdots,k-1$, and let $W_{(1)} \leqq \cdots \leqq W_{(p)}$ denote the ordered $W_1,\cdots,W_p$ with $p= k-1$. Then by considering the orders of W_i's first, we have the following identity:

$$\begin{aligned} &P_{\underline{\theta}}(Y_{(k-t+1)} > Y_{(k-t)} + c) \\ &= P_{\underline{\theta}}(Y_1 > W_{(p-t+1)}, \min(Y_1, W_{(p-t+2)}) > W_{(p-t+1)} + c) \\ &\quad + P_{\underline{\theta}}(Y_1 \leqq W_{(p-t+1)}), W_{(p-t+1)}) > \max(Y_1, W_{(p-t)}) + c) . \end{aligned}$$

It follows from this identity that the following identities hold:

$$\begin{aligned} &P_{\underline{\theta}}(Y_{(k-t+1)} > Y_{(k-t)} + c, \min_{p-t+2\leqq j\leqq p} W_j \geqq W_{p-t+1} \geqq \max_{1\leqq i\leqq p-t} W_i) \\ &= P_{\underline{\theta}}(\min_{p-t+2\leqq j\leqq p} (Y_1, W_j) > W_{p-t+1} + c, W_{p-t+1} > \max_{1\leqq i\leqq p-t} W_i) \\ &\quad + P_{\underline{\theta}}(\min_{p-t+2\leqq j\leqq p} W_j > W_{p-t+1} > \max_{1\leqq i\leqq p-t} (Y_1, W_i) + c) \\ &= \int_{-\infty}^{\infty} \bar{F}(y-\theta_1+c) \prod_{j=p-t+2}^{p} \bar{F}(y-\eta_j+c) \prod_{i=1}^{p-t} F(y-\eta_j) f(y-\eta_{p-t+1}) dy \\ &\quad + \int_{-\infty}^{\infty} F(y-\theta_1-c) \prod_{j=p-t+2}^{p} \bar{F}(y-\eta_j) \prod_{i=1}^{p-t} F(y-\eta_i-c) f(y-\eta_{p-t+1}) dy. \end{aligned}$$

By differentiating this expression w.r.t. θ_1, we have

$$\begin{aligned} &\frac{\partial}{\partial\theta_1} P_{\underline{\theta}}(Y_{(k-t+1)} > Y_{(k-t)} + c, \min_{p-t+2\leqq j\leqq p} W_j \geqq W_{p-t+1} \geqq \max_{1\leqq i\leqq p-t} W_i) \\ &= \int_{-\infty}^{\infty} \prod_{j=p-t+2}^{p} \bar{F}(y-\eta_j+c) \prod_{i=1}^{p-t} F(y-\eta_i) f(y-\theta_1+c) f(y-\eta_{p-t+1}) dy \\ &\quad - \int_{-\infty}^{\infty} \prod_{j=p-t+2}^{p} \bar{F}(y-\eta_j) \prod_{i=1}^{p-t} F(y-\eta_i-c) f(y-\theta_1-c) f(y-\eta_{p-t+1}) dy \\ &= \int_{-\infty}^{\infty} \prod_{j=p-t+2}^{p} \bar{F}(y-\eta_j) \prod_{i=1}^{p-t} F(y-\eta_i-c) \\ &\qquad [f(y-\theta_1) f(y-c-\eta_{p-t+1}) - f(y-c-\theta_1) f(y-\eta_{p-t+1})] dy \end{aligned}$$

Since the expression in brackets is non-positive by the MLR property of $f(y-\theta)$, it follows that the probability

$$P_{\underline{\theta}}(Y_{(k-t+1)} > Y_{(k-t)} + c, \min_{p-t+2\leqq j\leqq p} W_j \geqq W_{p-t+1} \geqq \max_{1\leqq i\leqq p-t} W_i)$$

is non-increasing in θ_1. Therefore the result regarding $\theta_{[1]}$ follows by observing that the ordering of W_i's can be replaced by any other permutation of W_i's. The monotonicity in $\theta_{[k]}$ can be proved in a similar fashion, and the proof is omitted. ■

To define a lower confidence bound on PCS, we consider an auxiliary function L introduced by Kim(1986), which is defined by

$$H(L(w)-w) + H(-L(w)-w) = \alpha \tag{2.2}$$

for $w \geqq x_{\alpha/2}$, where H is the cdf of $(Y_1 - \theta_1) - (Y_2 - \theta_2)$ and $x_{\alpha/2}$ is the upper $\alpha/2$ quantile of $H(x)$. Note that the function $L(w)$ is strictly increasing in $w \geqq x_{\alpha/2}$.

The next result presents a $100(1-\alpha)\%$ lower confidence bound on $\theta_{[k-t+1]} - \theta_{[k-t]}$, which is a generalization of a result in Kim (1986).

Theorem 1. Assume that $f(y-\theta)$ has the MLR property in y and θ. Then we have

$$\inf_{\underline{\theta}} P_{\underline{\theta}}(\theta_{[k-t+1]} - \theta_{[k-t]} \geqq L(Y_{(k-t+1)} - Y_{(k-t)})) = 1-\alpha \ ,$$

where $L(w)$ is defined by (2.2) for $w \geqq x_{\alpha/2}$ and 0 for $0 \leqq w \leqq x_{\alpha/2}$.

Proof. For any fixed $\theta_{[k-t+1]}$ and $\theta_{[k-t]}$, let $\triangle = \theta_{[k-t+1]} - \theta_{[k-t]}$. Then it follows from the repeated applications of Lemma 1 that for all $\underline{\theta}$

$$\begin{aligned} & P_{\underline{\theta}}(\triangle \geqq L(Y_{(k-t+1)} - Y_{(k-t)})) \\ & = P_{\underline{\theta}}(L^{-1}(\triangle) \geqq Y_{(k-t+1)} - Y_{(k-t)}) \\ & \geqq P_{\underline{\theta}}(L^{-1}(\triangle) \geqq | Y_{[k-t+1]} - Y_{[k-t]} |), \end{aligned}$$

where $L^{-1}(0)$ is defined as $x_{\alpha/2}$ and $Y_{[i]}$ corresponds to $\theta_{[i]} (i = k-t, k-t+1)$. The equality is obtained when $\theta_{[1]} = \cdots = \theta_{[k-t]} = -\infty$ and $\theta_{[k-t+2]} = \cdots = \theta_{[k]} = +\infty$. Furthermore, for any value of $\triangle$, we have

$$\begin{aligned} & P_{\underline{\theta}}(L^{-1}(\triangle) \geqq \ Y_{[k-t+1]} - Y_{[k-t]} \ ) \\ & = 1 - \{H(\triangle - L^{-1}(\triangle)) + H(-\triangle - L^{-1}(\triangle))\} \\ & = 1 - \alpha \end{aligned}$$

by the definition of L. Thus the proof is completed. ■

The next corollary follows from (2.1) and Theorem 1, and it provides a $100(1-\alpha)\%$ conservative lower confidence bound on PCS.

Corollary 1. Under the assumption of Theorem 1, we have

$$P_{\underline{\theta}}(PCS \geqq \hat{P}_L) \geqq 1 - \alpha \text{ for all } \underline{\theta}$$

where

$$\hat{P}_L = t \int_{-\infty}^{\infty} F^{k-t}(y+L(Y_{(k-t+1)}-Y_{(k-t)}))\, \bar{F}^{t-1}(y)f(y)dy.$$

3. Normal Means Problem

Let X_{ij} be independent normal random variables with mean $\mu_i (i=1,\cdots,k, j=1,\cdots,n)$ and a common variance σ^2. We consider the problem of selecting t best populstions correspo-nding to $\mu_{[k]},\cdots,\mu_{[k-t+1]}$ where $\mu_{[1]} \leqq \cdots \leqq \mu_{[k]}$ denote the ordered μ_i's. For the natural selection rule which selects the populations associated with the t largest sample means $X_{(k-t+1)},\cdots,X_{(k)}$, where $X_{(1)} \leqq \cdots \leqq X_{(k)}$ are the ordered sample means, the PCS in (1.1) is given by

$$PCS = \int_{-\infty}^{\infty} \prod_{i=1}^{k-t} \Phi(\sqrt{n}\,(y-\mu_{[i]})/\sigma)\, d[1-\prod_{j=k-t+1}^{k} \bar{\Phi}(\sqrt{n}\,(y-\mu_{[j]})/\sigma)]$$

where Φ is the cdf of the standard normal distribution and $\bar{\Phi} = 1-\Phi$.

When the common variance is unknown, let $\hat{\sigma}^2$ denote the pooled sample variance. Then, by considering the conditional coverage probability given the sample variance $\hat{\sigma}^2$ and by taking $Y_i = \sqrt{n}\,X_i/\sigma$, $\theta_i = \sqrt{n}\,\mu_i/\sigma$ in Corollary 1, the following $100(1-\alpha)\%$ confidence statement can be made:

$$PCS \geqq t \int_{-\infty}^{\infty} \Phi^{k-t}[x+\sqrt{2}\, h_\nu(\sqrt{n}\,(\bar{X}_{(k-t+1)}-\bar{X}_{(k-t)})/\sqrt{2}\,\hat{\sigma}]\, \bar{\Phi}^{t-1}(x)\, d\Phi(x) \tag{3.1}$$

where $h_\nu(w)$ is given by

$$\int_0^{\infty}[\Phi(h_\nu(w)-wu)+\Phi(-h_\nu(w)-wu)]\, dQ_\nu(u) = \alpha \tag{3.2}$$

for $w \geqq t_{\alpha/2}(\nu)$ and 0 for $0 \leqq w \leqq t_{\alpha/2}(\nu)$. Here $t_{\alpha/2}(\nu)$ is the upper $\alpha/2$ quantile of t-distribution with $\nu = k(n-1)$ degrees of freedom and Q_ν is the cdf of $\hat{\sigma}/\sigma$.

As noted in Section 2, the function h_ν in (3.2) was tabulated in Kim(1986). Thus after finding the values of h_ν from the tables in Kim, the lower confidence bound in (3.1) can be obtained from the tables in Bechhofer (1954) for the integral value.

4. Normal Variances Problem

In practice it is often of interest to select the populations with small variances. This section treats the problem of selecting the populations with t smallest variances out of k independent normal populations.

Assuming equal sample sizes for each population, let S_i^2 denote the sample variance so that $(n-1)S_i^2/\sigma_i^2 (i=1,\cdots,k)$ has a chi-square distribution with (n-1) degrees of fredom. In this case, the PCS in (1.1) for the natural rule, which selects the populations associated with t smallest sample variances, is given by

$$PCS= \int_0^\infty \prod_{i=1}^{t} G(y/\sigma^2_{[i]})\, d[1-\prod_{j=t+1}^{k} \bar{G}(y/\sigma^2_{[j]})] \tag{4.1}$$

where $G = G_{n-1}$ is the cdf of chi-square distribution with n-1 degrees of freedom and $\bar{G} = 1-G$. In (4.1), $\sigma^2_{[1]}\leqq\cdots\leqq\sigma^2_{[k]}$ denote the ordered variances $\sigma_1^2,\cdots,\sigma_k^2$.

Methods similar to those in Section 2 yield that the $100(1-\alpha)\%$ lower confidence bound on the PCS in (4.1) is given by

$$\hat{P}_L= \int_0^\infty G^t[yd_\nu(S^2_{(t+1)}/S^2_{(t)})]d[1-\bar{G}^{k-t}(y)] \tag{4.2}$$

where $S^2_{(1)} \leqq\cdots\leqq S^2_{(k)}$ are the ordered sample variances. The function $d_\nu(w)$ in (4.2) is defined by

$$F_\nu(d_\nu(w)/w)+ F_\nu(wd_\nu(w))^{-1})= \alpha \tag{4.3}$$

for $w \geqq F_{\alpha/2}(\nu,\nu)$ and 1 for $0 < w < F_{\alpha/2}(\nu,\nu)$. Here $F_\nu(y)$ is the cdf of F-distribution with $\nu=$ n-1 and $\nu=$ n-1 degrees of freedom, and $F_{\alpha/2}(\nu,\nu)$ is the upper $\alpha/2$ quantile of $F_\nu(y)$.

The vaues of the function $d_\nu(w)$ defined by (4.3) are given in Table 1 for $\alpha=$ 0.05, 0.10, and for selected values of ν and $w \geqq F_{\alpha/2}(\nu,\nu)$. It can be easily observed from (4.3) that

$$\lim d_\nu(w)/w= F_{1-\alpha}(\nu,\nu) \text{ and}$$

$$F_{1-\alpha/2}(\nu,\nu)< d_\nu(w)/w<F_{1-\alpha}(\nu,\nu) \text{ for } w > F_{\alpha/2}(\nu,\nu).$$

This fact was utilized in computing values of $d_\nu(w)$. For selected cases, the shapes of $d_\nu(w)$ are given Figure 1.

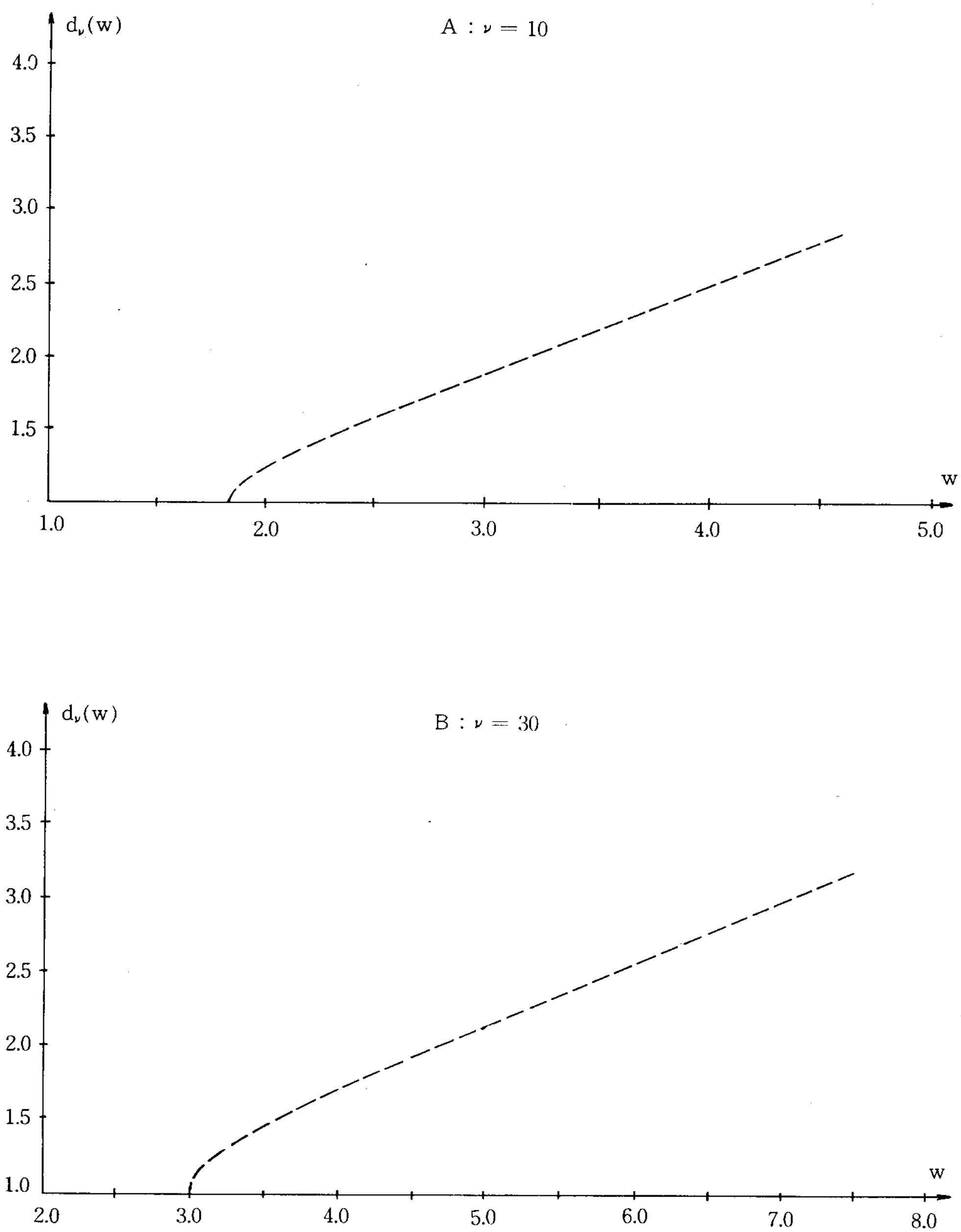

Figure 1. The ralues of $d\nu(w)$ for $\alpha = 0.10$

Table 1. The values of $d_\nu(w)$ vs $c = w / F_{\alpha/2}(\nu,\nu)$ for $\alpha = 0.10,\ 0.15$

	$\alpha = 0.10$					$\alpha = 0.05$				
c \ ν	10	15	20	25	30	10	15	20	25	30
1.05	1.222	1.198	1.182	1.172	1.163	1.206	1.183	1.168	1.159	1.151
1.10	1.328	1.292	1.269	1.253	1.242	1.305	1.271	1.250	1.236	1.225
1.15	1.417	1.370	1.342	1.323	1.308	1.388	1.345	1.319	1.302	1.289
1.20	1.497	1.442	1.409	1.387	1.370	1.464	1.413	1.384	1.363	1.349
1.25	1.572	1.510	1.473	1.448	1.430	1.535	1.478	1.445	1.423	1.407
1.30	1.644	1.575	1.535	1.508	1.488	1.604	1.541	1.505	1.481	1.464
1.35	1.713	1.639	1.596	1.567	1.546	1.671	1.603	1.564	1.539	1.521
1.40	1.782	1.702	1.656	1.626	1.604	1.737	1.664	1.623	1.596	1.577
1.45	1.849	1.764	1.716	1.684	1.661	1.802	1.725	1.682	1.654	1.634
1.50	1.915	1.826	1.775	1.742	1.719	1.866	1.785	1.740	1.711	1.690
1.60	2.047	1.949	1.894	1.859	1.833	1.993	1.905	1.856	1.825	1.803
1.70	2.177	2.071	2.031	1.975	1.948	2.119	2.024	1.972	1.939	1.915
1.80	2.306	2.193	2.131	1.091	2.063	2.245	2.143	2.088	2.053	2.028
1.90	2.435	2.315	2.250	2.207	2.177	2.370	2.262	2.204	2.167	2.241
2.00	2.564	2.437	2.368	2.324	2.292	2.495	2.382	2.320	2.281	2.253
2.10	2.692	2.559	2.487	2.440	2.406	2.620	2.501	2.436	2.395	2.366
2.20	2.821	2.681	2.605	2.556	2.551	2.745	2.620	2.552	2.509	2.479
2.30	2.949	2.803	2.723	2.672	2.636	2.870	2.739	2.669	2.623	2.591
2.40	3.077	2.925	2.842	2.788	2.750	2.995	2.858	2.785	2.737	2.704
2.50	3.206	3.047	2.960	2.905	2.865	3.120	2.977	2.901	2.851	2.816

After finding the values of $d_\nu(w)$, we need the integral value in (4.2) to obtain the lower confidence bound P_L. Table 2 at the end of this article provides the values of

$$P_L(d) = \int_0^\infty G^t(dy)d[1-\bar{G}^{k-t}(y)] \tag{4.4}$$

for selected values of ν and $d > 1$, and for $k = 3,5$. Some other cases have been computed, but are not reported here. Integral in (4.4) reduces to that in Gupta and Sobel (1962) when $t = 1$.

The values of $d_\nu(w)$ in Table 1 were found numerically by finding a root of (4.3) via the bisection method with the accuracy up to 10^{-5}. The values of the cdf of F-distribution were obtained by IMSL's subroutine MDFD. For constructing Table 2, evaluation of the integral in (4.4) was done by using IMSL subroutine MDCH and 32 points Gauss-Laguerre quardrature in the IBM Scientific Subroutine Package.

Finally, it should be remarked that the result in this section holds for scale parameters families under the assumption of MLR property of the pdf $\theta_i^{-1} f(y/\theta_i)$ in y and $\theta_i (i = 1,\cdots,k)$.

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Table 2. The values of $P_L(d)$ for $k = 3,5$

	$k=3$ $t=1$					$k=3$ $t=2$				
c \ ν	10	15	20	25	30	10	15	20	25	30
1.1	.391	.406	.418	.429	.439	.397	.411	.424	.424	.444
1.2	.446	.475	.499	.520	.539	.456	.485	.509	.530	.549
1.3	.498	.539	.574	.604	.630	.512	.553	.586	.616	.641
1.4	.546	.598	.641	.677	.708	.562	.614	.655	.690	.720
1.5	.590	.651	.639	.739	.773	.608	.667	.714	.752	.784
1.6	.630	.698	.750	.791	.825	.650	.714	.764	.803	.835
1.7	.667	.739	.793	.834	.866	.687	.755	.806	.844	.874
1.8	.700	.775	.829	.863	.898	.720	.790	.841	.877	.905
1.9	.729	.806	.859	.896	.923	.749	.821	.869	.904	.929
2.0	.756	.833	.884	.918	.942	.775	.846	.893	.924	.946
2.1	.780	.856	.904	.936	.957	.799	.868	.912	.941	.960
2.2	.802	.876	.921	.949	.967	.819	.887	.923	.953	.970
2.3	.821	.893	.925	.960	.975	.837	.903	.941	.963	.977
2.4	.838	.908	.947	.969	.981	.954	.917	.951	.971	.983
2.5	.853	.921	.956	.975	.986	.868	.928	.960	.977	.987
2.6	.867	.931	.964	.981	.989	.881	.938	.967	.982	.990
2.7	.880	.940	.970	.985	.992	.892	.946	.973	.986	.993
2.8	.891	.948	.975	.988	.994	.903	.953	.977	.989	.994
2.9	.901	.955	.979	.990	.995	.912	.960	.981	.991	.996
3.0	.910	.961	.983	.992	.996	.920	.963	.984	.993	.997
3.1	.918	.966	.986	.994	.997	.927	.969	.987	.994	.997
3.2	.925	.970	.988	.995	.998	.934	.973	.989	.995	.998
3.3	.931	.974	.990	.996	.998	.939	.977	.991	.996	.999
3.4	.937	.977	.992	.997	.999	.945	.980	.992	.997	.999
3.5	.943	.980	.993	.997	.999	.949	.982	.993	.998	.999
3.6	.947	.982	.994	.998	.999	.954	.984	.994	.998	.999
3.7	.952	.985	.995	.998	.999	.958	.986	.995	.998	.999
3.8	.956	.986	.996	.999	.999	.961	.988	.995	.999	.999
3.9	.959	.988	.996	.999	.999	.964	.989	.997	.999	.999
4.0	.962	.989	.997	.999	.999	.967	.990	.997	.999	.999

Table 2. (continued)

	k = 5 t = 1					k = 5 t = 2				
c \ ν	10	15	20	25	30	10	15	20	25	30
1.1	.246	.259	.271	.281	.290	.137	.148	.157	.163	.172
1.2	.294	.321	.345	.366	.386	.173	.202	.224	.244	.252
1.3	.341	.383	.420	.452	.482	.223	.263	.298	.330	.360
1.4	.388	.444	.492	.534	.572	.270	.325	.374	.418	.460
1.5	.433	.502	.560	.609	.653	.316	.387	.449	.504	.554
1.6	.476	.556	.622	.676	.722	.362	.447	.520	.583	.638
1.7	.517	.606	.677	.734	.781	.407	.505	.586	.634	.711
1.8	.556	.652	.726	.784	.829	.451	.559	.646	.716	.772
1.9	.592	.694	.769	.825	.867	.492	.609	.699	.768	.822
2.0	.625	.731	.806	.859	.898	.531	.654	.745	.812	.862
2.1	.656	.764	.837	.887	.922	.568	.695	.785	.849	.894
2.2	.685	.793	.863	.910	.940	.603	.731	.819	.878	.918
2.3	.711	.819	.886	.928	.954	.634	.764	.848	.903	.938
2.4	.935	.841	.905	.942	.965	.664	.793	.873	.922	.952
2.5	.757	.861	.920	.954	.974	.691	.818	.893	.938	.964
2.6	.777	.878	.933	.963	.980	.716	.840	.911	.950	.972
2.7	.796	.894	.944	.971	.985	.739	.860	.925	.960	.979
2.8	.813	.907	.953	.977	.988	.760	.877	.937	.968	.984
2.9	.828	.918	.961	.981	.991	.780	.892	.947	.974	.987
3.0	.842	.928	.967	.985	.993	.797	.905	.956	.979	.990
3.1	.855	.937	.973	.988	.995	.814	.916	.963	.983	.993
3.2	.867	.945	.977	.990	.996	.828	.926	.968	.987	.994
3.3	.877	.951	.981	.992	.997	.842	.935	.974	.989	.995
3.4	.887	.957	.984	.994	.998	.854	.943	.978	.991	.997
3.5	.896	.962	.986	.995	.998	.866	.949	.981	.993	.997
3.6	.904	.967	.988	.996	.999	.876	.955	.984	.994	.993
3.7	.911	.970	.990	.997	.999	.886	.960	.986	.995	.993
3.8	.918	.974	.992	.997	.999	.894	.963	.988	.996	.999
3.9	.924	.977	.993	.998	.999	.902	.968	.990	.997	.999
4.0	.930	.979	.994	.998	.999	.909	.972	.992	.997	.999

Table 2. (continued)

	k = 5 t = 3					k = 5 t = 4				
c \ ν	10	15	20	25	30	10	15	20	25	30
1.1	.140	.151	.160	.168	.176	.257	.270	.232	.292	.301
1.2	.185	.209	.230	.250	.268	.315	.342	.366	.388	.407
1.3	.232	.272	.307	.329	.370	.372	.414	.450	.481	.510
1.4	.281	.337	.386	.430	.471	.426	.481	.528	.568	.604
1.5	.330	.401	.462	.517	.565	.578	.544	.598	.645	.685
1.6	.379	.468	.585	.597	.950	.526	.601	.661	.711	.808
1.7	.426	.522	.601	.667	.722	.570	.652	.716	.767	.808
1.8	.470	.576	.660	.728	.782	.610	.697	.763	.813	.852
1.9	.513	.626	.712	.779	.830	.647	.737	.801	.850	.886
2.0	.552	.671	.758	.822	.869	.681	.772	.836	.881	.913
2.1	.589	.711	.796	.857	.899	.711	.802	.864	.905	.934
2.2	.623	.746	.829	.885	.923	.738	.829	.887	.925	.950
2.3	.655	.778	.857	.908	.941	.963	.851	.906	.940	.962
2.4	.684	.805	.880	.927	.955	.985	.871	.922	.952	.971
2.5	.711	.830	.900	.941	.966	.805	.888	.935	.962	.978
2.6	.735	.851	.916	.953	.974	.823	.903	.946	.970	.983
2.7	.757	.869	.930	.962	.980	.839	.915	.955	.976	.987
2.8	.777	.886	.941	.970	.985	.853	.926	.962	.981	.990
2.9	.796	.900	.951	.976	.988	.866	.936	.969	.985	.992
3.0	.813	.912	.959	.981	.991	.878	.944	.974	.988	.994
3.1	.829	.923	.965	.984	.993	.898	.957	.981	.992	.997
3.2	.842	.932	.971	.987	.995	.898	.951	.973	.990	.996
3.3	.855	.940	.975	.990	.996	.907	.962	.984	.994	.997
3.4	.866	.947	.979	.992	.997	.914	.967	.987	.995	.998
3.5	.877	.953	.982	.993	.997	.921	.971	.989	.996	.998
3.6	.886	.829	.985	.995	.998	.928	.974	.991	.994	.999
3.7	.895	.963	.987	.996	.998	.934	.977	.992	.997	.999
3.8	.903	.968	.989	.996	.999	.939	.980	.993	.998	.999
3.9	.911	.971	.991	.997	.999	.944	.982	.994	.998	.999
4.0	.917	.974	.992	.998	.999	.948	.984	.995	.998	.999