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Existence Proffs of a Nash Equilibrium to a General Class of Differential Games

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미분게임 일반 모형에 대한 Nash 균형해의 존재증명

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Abstract

This paper extends the existence proofs of a Nash equilibrium to a more general class of differential game models with constraints on the control spaces. With the assumptions of continuity, convexity, and compactness, the existence is proved using Kakutani Theorem and via a path-following approach. Furthermore, the proof for a period-by-period optimization of multi-period problems provides an insight to a numerical solution algorithm to differential game models with constraints.

1. Introduction

The differential game is a mathematical decision-aid tool in a conflict situation which evolves over time. The model has been applied widely in managerial problems including investment, production, marketing. Feichtinger and Jorgensen[2] provides an excellent survey for applications of the differential game model. In order to apply the theory to the analysis of economic competition, where mutual interests play a significant role, nonzero-sum formulations are mostly appropriate. Zero-sum game models usually rule out the possibility of mutual benefits between the conflicting parties(Ciletti and Starr[1]).

Although necessary conditions for an optimal solution of the differential game problems may be derived by an application

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of the maximum principle of optimal control theory, it is, in general, very difficult to find an analytical solution for the necesary conditions. The set of necessary conditions of an optimal solution to differential gme problems requires solving a system of differential equations. Solution methods and existence proofs for simple differential games are found in Starr and Ho[11], Friedman[3] and Neese and Pindyck[6], Simman and Takayama[10] gives an example for a single state linear-quadratic game with constraints. For open-loop nonzero-sum differential games, Scalzo[8] proved existence for any finite duration. Scalzo's work has been extended by Wilson[13] and Williams[12] to games with incomplete information and by Scalzo and Williams[9] to games with nonlinear state equations. All three extensions dealt with the finite horizon case.

With the assumptions of continuity, convexity, and compactness, this paper shows the existence of a Nash equilibrium for a more general class of differential games. To prove it, the Kakutani theorem is applied after discretization of the problem. The rationale for a period-by-period operation of multi-period problems is provided, using a dynamic programming approach. In conjunction with development of a numerical solution algorithm to a general class of differential game models with nonlinear constraints, another way of existence proof is provided using a path-following approach.

2. Differential Game Problem

We first define a general type of the nonzero-sum two-person differential game. Let subscript t denote time, and superscript i on functions and subscript for variables or parameters stand for player i. The state variable at time t, $z_t=(z_{1t}, z_{2t})$, is governed by a system of the first order differential equations,

$$\dot{z}_{it}=g^i(z_t, u_t),\quad i=1, 2. \quad \cdots\cdots\cdots\cdots\cdots\cdots \quad (1)$$

where $u_t=(u_{1t}, u_{2t})$ are the control variables. The initial states at time 0 are assumed to be known. The players set the control variables to achieve a desired state. They may have a limitation in setting the control variables. We assume the realized states at time t determine the control spaces for each player,

$$(u_{it})\in\Omega^i(t, z_t),\ i=1, 2. \quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots \quad (2)$$

A strategy, (u_t) is admissible if it belongs to the space defined by the incumbent state, z_t.

The game begins at some initial time and state $(0, z_0)$, and terminates at (T, z_T). The terminal time T can be chosen freely. If T is a pre-specified point in time, the payoff to player i over the finite horizon is given by

$$J^i(z_t, u_t)=S^i(T, z_T)+\int_0^T e^{-rt}f^i(t, z_t, u_t)dt,$$

$i=1, 2.$ .... (3)

where S^i is a real valued function representing player i's salvage value at T if the terminal state of the game is z_T, and f^i is a real valued function on (t, z, u)-space. The functions S^i, f^i are also assumed to be of class C^3 with respect to their own arguments. The objective for each player is to select a control strategy which maximizes J^i.

In (2), we denoted the set of admissible values for the control variables by $\Omega^i(t, z_t)$. For the problem to be defined in a more manageable setting, we shall further assume that the control variables must satisfy the following constratints,

$h^i(t, z_t, u_t) \geq 0, i=1, 2.$ .... (4)

where h^i is a real-valued, third order differential function with respect to all the arguments. The admissible control spaces are now assumed to be expressed by a form of (4).

3. Existence Theorems

The problem described in the previous section can be writen in a discretized form as follows,

$$(P_i)\quad \underset{\{u_i\}}{\text{Max}}\ J^i=\Sigma_{t=0}^{T-1}(1+r)^{-t}f^i(z_t, u_t)+(1+r)^{-T}S^i(z_T)$$

subject to,

$$z_{j,t+1}-z_{jt}=g^j(z_t, u_t),\ j=1, 2,\ t=0,\cdots, T-1$$

$$h^i(z_t, u_t)\geq 0,\ t=0,\cdots,T-1.$$

The problem is now to find a sequence of equilibrium control vectors, $\{u_t\}_{t=0}^{T-1}$, and hence a sequence of the state vectors, $\{z_t\}_{t=1}^{T}$, which maximizes the objective function. Define a function W^i as

$$W^i(z_t, u_t,\cdots,u_{T-1})=\Sigma_{\tau=t}^{T-1}(1+r)^{-\tau}f^i(z_\tau, u_\tau)+(1+r)^{-T}S^i(z_T)$$

and

$$V^i(z_t, t)=\underset{\{u_{it}\}}{\text{Max}}\ (1+r)^{-t}f^i(z_t, u_t)+V^i[z_{t+1}(z_t, u_t), t+1]$$

(SP_{it}) subject to,

$$z_{j,t+1}-z_{jt}=g^j(z_t, u_t),\ j=1, 2$$

$$h^i(z_t, u_t)\geq 0.$$

$V^i(z_t, t)$ represents the optimal objective value from t to the terminal time period if the incumbent state at t is z_t. We write $z_{t+1}(z_t, u_t)$ to show explicitly that z_{t+1} depends on z_t and u_t. The subproblem of player i at time t, (SP_{it}), is to maximize his payoffs from time t to T for z_t given. Note $V^i(., T)=(1+r)^{-T}S^i(z_T)$.

The following theorem provides us with a basis for the claim that a period-by-period solution consists of a Nash equilibrium for the original problem.

Theorem 1. If u_t^* is a Nash equilibrium to (SP_{it}), for $t=0,\cdots,T-1$, then $\{u_t^*\}_{t=0}^{T-1}$ is a Nash equilibrium to the problem(P). Therefore, if (SP_{it}) has an equilibrium solution due to Nash for all the subperiods t, there exists a Nash equilibrium point to the problem(P).

Proof. At $t=T-1$, $V^i[z_T(z_{T-1}, u_{T-1}), T]=(1+r)^{-T}S^i(z_T)$. Since u^*_{T-1} is a Nash equilibrium to $(SP_{i,T-1})$ for a given feasible z_{T-1},

$$(1+r)^{-(T-1)}f^i(z_{T-1}, u^*_{T-1})+V^i[z_T(z_{T-1}, u^*_{T-1}), T]\geq W^i(z_{T-1}, u^{T-1})$$

where $u_{T-1}=(u_{i,T-1}, u^*_{-i,T-1})$, i.e., any admissible strategy for player i while the other player's Nash equilibrium policy remains fixed. From now on, we denote u_t for$(u_{it}, u^*_{-i,t})$. At any time,

$$\begin{aligned}&(1+r)^{-t}f^i(z_t, u^*_t)+V^i[z_{t+1}(z_t, u^*_t), t+1]\\ \geq&(1+r)^{-t}f^i(z_t, u_t)+V^i[z_{t+1}(z_t, u_t), t+1]\\ \geq&(1+r)^{-t}f^i(z_t, u_t)+W^I(z_{t+1}(z_t, u_t), u_{t+1},\cdots, u_{T-1})\\ =&W^i(z_t, u_t, u_{t+1},\cdots,u_{T-1}).\end{aligned}$$

But we know that

$$\begin{aligned}&(1+r)^{-t}f^i(z_t, u^*_t)+V^i[z_{t+1}(z_t, u^*_t), t+1]\\ =&(1+r)^{-t}f^i(z_t, u^*_t)+W^i[z_{t+1}(z_t, u^*_t), u^*_{t+1},\cdots,u^*_{T-1}]\\ =&W^i(z_t, u^*_t, u^*_{t+1},\cdots,u^*_{T-1}].\end{aligned}$$

This implies that $W^i(z_t, u^*_t,\cdots,u^*_{T-1}]\geq W^i(z_t, u_t, u_{t+1},\cdots,u_{T-1})$. In other words, a strategy, $\{u^*_{i\tau}\}^{T-1}_{\tau=t}$, provides player i with the maximum payoff if the other player chooses a strategy of $\{u^*_{-i,\tau}\}^{T-1}_{\tau=t}$. This implies $\{u^*_\tau\}^{T-1}_{\tau=t}$, consists of a Nash equilibrium policy to (P). Q.E.D.

Theorem 2. (SP_{it}) has a Nash equilibrium if(i) f^i and S^i are continuous in u_t, and concave in u_{it}, (ii) g^j, $j=1, 2$, is linear in u_{it}, and the Hessian matrix of V^i with respect to z_{t+1} is negative semidefinite, and (iii) the functions h^i are convex in u_{it}, $i=1, 2$, and $D_t=\{(z_t, u_t)\mid h^i(z_t, u_t)\geq 0\}$ is a nonempty compact convex set.

Proof. The Kakutani Theorem is applied. Since f^i and S^i are continuous, the V^i is also a continuous mapping. The condition(ii) guarantees V^i to be concave in u_{it}. Therefore, the objective function of (SP_{it}) is continuous and concave in u_{it}. Define a point-to-set mapping $F^i(u_t)$ as

$$F^i(u_t)=\{u^*_{it}\mid u^*_{it} \text{ is optimal for } (SP_{it}) \text{ for given } u_{-i,t}\},$$

and, $F(u_t)=F^1\times F^2=\{u^*_t\}$, where u^*_t is such that u^*_{it} maximizes(SP_{it}) for given $u_{-i,t}$, $i=1, 2$. Since the objective functions are concave in their own control variables and D_t is convex, the set $F^i(u_t)$ is convex for any $u_t\in D_t$. To show this, suppose $u^1_{it}, u^2_{it}\in F^i(u_t)$. Consider $\tilde{u}_{it}=\beta u^1_{it}+(1-\beta)u^2_{it}$ for $0\leq\beta\leq 1$. Then, $\tilde{u}_{it}\in D_t$ by convexity of D_t, i.e., $\tilde{u}_{it}$ is feasible. Furthermore, by convexity of the objective function, $\tilde{u}_{it}$ should be optimal. That is, $\tilde{u}_{it}\in F^i(u_t)$. Therefore, F is convex since the Cartesian product of convex sets is also convex. From the compactness of D_t and continuity of the objective functions, we can show that F is an upper hemicontinuous mapping that maps each point of the convex, compact set D_t into a closed convex subset of D_t[1]. Then, the Kakutani

Theorem says that there exists a fixed pointe $u^*_t \in D_t$ such that $u^*_t = F(u^*_t)$. By the definition of the function F, the fixed point u^*_t is a Nash equilibrium, Q.E.D.

With the assumptions of Theroem 2, it is also possible to show the existence of an equilibrium to the partially cooperative 3-person differential games(see Kim[5]). However, the dynamic programming approach above mentioned is hardly implementable. For purposes of computation, consider another way of existence proof. Recall the Hamiltonian equation is defined by adjoining the state equation with the payoff function at time t, using the adjoint variables[2]. Once we discretize the problem, the discrete Hamiltonian equation is written as follows,

$$H^i(z_t, u_t, \lambda_{i,t+1}) = f^i(z_t, u_t) + \Sigma_j \lambda_{ij,t+1} g^j(z_t, u_t), \quad i, j = 1, 2.$$

At time t, we are concerned with the contribution of z_{t+1} to the objective function. Thus, the adjoint variable has a time subscript (t+1) in the above.

Theorem 3. Assume V^i is continuously differentiable with respect to all the arguments. Define a Hamiltonian maximization problem as

$$(SP_{it})' \quad \begin{array}{l} \text{Max } H^i(z_t, u_t, \lambda_{i,t+1}) \\ \{u_{it}\} \\ \text{subject to,} \\ h^i(z_t, u_t) \geq 0. \end{array}$$

Then, $(SP_{it})'$ is a first order Taylor approximation of (SP_{it}). if He

Proof. Since V^i is continuously differentiable, we can use the Taylor expansion to write,

$$V^i[z_{t+1}(z_t, u_t), t+1] = V^i(z_t, t) + (\partial V^i/\partial z_{t+1})^\tau (z_{t+1} - z_t) + \partial V^i/\partial t.$$

Since the first and last terms are independent of u_t, and since $z_{t+1} - z_t = g(z_t, u_t)$, the maximand of (SP_{it}) is approximately equivalent to $H^i(z_t, u_t, \lambda_{i,t+1})$. Q.E.D.

From Theorem 1, we have seen that the period-by-period equilibrium solutions consist of a Nash equilibrium to the combined problem, (P). And, Theorem 3 states that (SP_{it}) and $(SP_{it})'$ are equivalent as the time interval gets small enough. Therefore, it suffices to show that $(SP_{it})'$ has an equilibrium in order to prove the existence of a Nash equilibrium to(P). The following Theorem provides the proof via a path-following approach.

Theorem 4. $(SP_{it})'$ has an equilibrium

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1) Garcia and Zangwill[4] proved upper hemicontinuity for an economic equilibrium problem. Based on their work, the upper hemicontinuous property of the mapping, F, for our problem is proved in the Appendix. For the proof, continuity and compactness should be assumed.

2) If we use the definition of the valued function, the adjoint variable λ_{ijt} is writen as $\lambda_{ijt} = \partial V^i(z_t, t)/\partial z_{jt}$, i, j=1, 2.

point for given z_t and λ_{t+1}, if (i) the functions f^i and S^i are concave, h^i convex, and g^j, j=1, 2, linear on u_{it}, (ii) all functions are of class C^3, (iii) the constraint set has an interior pointe, is compact and convex, and (iv) regularity holds[3)].

Proof. The conditions(i) and (iii) assure that the Kuhn-Tucker conditions are necessary and sufficient conditions for an optimal solution. Supposed that x stands for a vector of all the primal and dual variables, and F(x) the Kuhn-Tucker equations to both players. For a proof, it suffices to show that there is a point satisfying the Kuhn-Tucker equations. Let us define a homotopy function as:

$$H(x, \tau)=(1-\tau)(x-x^{\circ})+\tau F(x),$$

where x° is an interior point, and τ is a path parameter such that $0\leqq\tau\leqq 1$. That is, $H(x, 0)=x-x^{\circ}$ and $H(x, 1)=F(x)$. Therefore we show that $H(x, 1)=0$ has a solution in order to show that $F(x)=0$ has a solution. With (ii), the homotopy function, $H(x, \tau)$, constructed based on the set of K-T conditions is of class C^2. Thus, if the conditions (ii) and (iv) hold, the implicit function theorem tells us that a continuously differentiable path exists in H. Since (iii) assures the existence of an interior point, we have a feasible point at $\tau=0$ where we start to follow the path. The condition (iii) implies that the path cannot drift to infinity, and it must reach a point such that $H(x, 1)=0$. By the construction of the homotopy function, the point satisfies the K-T conditions of the equilibrium programming problem. Q.E.D.

4. Discussion

Friedman[3] discussed existence of an equilibrium strategy. He assumed convexity and bounded control spaces to show existence of an open-loop solution to a differential game problem without constraints. For linear-quadratic games, he also proved existence of a closed-loop solution. In this paper, existence of a closed-loop Nash equilibrium has been proved to a more general class of differential games with constraints on control variables. Although the assumptions look very tight, most of the differentiable concave problems satisfy them. The most severe assumption is the linearity of the state equation on the own control variable.

The dynamic programming approach applied in the above Theorem 2 is hardly implementable for the purpose of computation. It is very difficult to solve (SP_{it}) directly. Notice, however, the problem transformed into a Hamiltonian maximization,

3) Here, we mean that the gradient matrix of a homotopy function constructed based on the Kuhn-Tucker conditions defined in the following proof, is invertible.

(SP_{it}), is much easier to solve if initial conditions are given. In this way, it is possible to desifgn a solution algorithm to problems with nonlinear constraints. The above theorems provide a basis for such an algorithm.

Appendix

Theorem A.1. Suppose the constraints for both Player 1 and Player 2 make a compact set, D_t, and the objective function, f^i, $i=1, 2$, is continuous. Then, the point-to-set mapping, F, is upper hemicontinuous.

Proof. To prove the theorem, we must show that given a pair of the sets of infinite points in D_t, $\{(u_1^k, u_2^k)\}_{k=1}^{\infty}$ and $\{(\tilde{u}_1^k, \tilde{u}_2^k)\}_{k=1}^{\infty}$ such that

$$(\tilde{u}_1^k, \tilde{u}_2^k) \in F(u_1^k, u_2^k) \quad \cdots\cdots (A.1)$$

and converging to their limit points, (u_1^∞, u_2^∞), then

$$(\tilde{u}_1^\infty, \tilde{u}_2^\infty) \in F(u_1^\infty, u_2^\infty). \quad \cdots\cdots (A.2)$$

Since D_t is compact, there exists a sequence of points, $\{(u_1^k, u_2^k)\}_{k=1}^{\infty}$ which converges to a point$(u_1^\infty, u_2^\infty) \in D_t$. When we take limits, the compactness of D_t implies that

$$(\tilde{u}_1^\infty, \tilde{u}_2^\infty) \in D_t \text{ and } (\tilde{u}_1^\infty, \tilde{u}_2^\infty) \in D_t \quad \cdots\cdots (A.3)$$

because a compact set contains all of its limit points. Thus, $\{\tilde{u}_1^\infty, \tilde{u}_2^\infty\}$ is feasible for $\{u_1^\infty, u_2^\infty\}$ given. To show optimality of the limit point, let us take another sequence of points, $\{(u_1^{*k}, u_2^k)\}_{k=1}^{\infty}$, converging to (u_1^*, u_2^∞) where $(u_1^{*k}, u_2^k) \in D_t$ for every k. And suppose u_1^* is optimal for Player 1 given u_2^∞. Then we have

$$f^1(\tilde{u}_1^\infty, u_2^\infty) \leqq f^1(u_1^*, u_2^\infty) \quad \cdots\cdots (A.4)$$

from (A.3). Since $\tilde{u}_1^k$ is optimal given u_2^k, and u_1^{*k} is feasible, we have

$$f^1(u_1^{*k}, u_2^k) \leqq f^1(\tilde{u}_1^k, u_2^k). \quad \cdots\cdots (A.5)$$

When we take limits on both sides of (A.5), we get

$$f^1(u_1^*, u_2^\infty) \leqq f^1(\tilde{u}_1^\infty, u_2^\infty) \quad \cdots\cdots (A.6)$$

since f^1 is a continuous function. (A.4) and (A.6) together imply that $\tilde{u}_1^\infty$ is optimal for Player 1 given u_2^∞. Simmilarly, we can show $\tilde{u}_2^\infty$ is optimal for Player 2 given u_1^∞. That is, (A.2) holds.

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