Journal of the Korean
OR/MS Society
Vol. 14, No. 2, Dec., 1989

Optimal Stocking for Age-replaced Non-repairable Items

Young-Taek Park* and Kyung-Soo Park**

修理 不可能한 品目의 壽命交換을 위한 最適 在庫政策

朴永宅*・朴景洙*

Abstract

Joint stocking and preventive age replacement policy is considered for non-repairable items assuming instantaneous replenishment. A recursive relationship among the optimal preventive replacement ages is obtained, which shows that the preventive replacement ages in a replenishment cycle form an increasing sequence due to the inventory carrying cost. Using this relationship, a procedure is given for determining how many units to purchase on each order and when to replace each unit after it has begun operating so as to minimize the total cost per unit time over an infinite time span. The problem can be simplified if equal preventive replacement ages are assumed, and the solution is very close to that of the original unconstrained problem.

1. Introduction

Maintenance policies for systems that are subject to stochastic failures have been treated extensively in the literature[7, 14, 16]. But most of the published research results have assumed that each time a unit is to be replaced, a new unit must be purchased and thus a procurement cost is incurred in every replacement. Procurement cost includes the cost of placing an order, buying, delivering, and receiving.

In many situations planned preventive replacement is more economical than unplanned corrective replacement resulting from the failure of a unit during operation.

* Department of Industrial Engineering, Sung-kyun-kwan University
** Department of Industrial Science, KAIST

As is usually the case, if the unit's failure chance increases with the time used, planned preventive replacement is warranted after some period of use.

In a preventive age replacement policy discussed by Barlow and Proschan[1], a unit is to be replaced by a new one if it has survived a certain age T(preventive replacement) or it has failed(corrective replacement), whichever occurs first. The planned preventive replacement age T is the only decision variable to be determined.

Instead of one for one ordering, consider the case(such as the provisioning of a non-repairable vital part) where more than one unit can be purchased at a given time. This might be desirable because of (i) fixed ordering cost which is independent of the quantity ordered, and (ii) an economy of scale obtained by ordering in batches(quantity discounts). Thus for the economical implementation of maintenance policies, we must determine when to place orders, how many units to purchase on each order and when to replace each unit after it has begun operating. This problem has similar atrributes as classical inventory problems and classical equipment replacement problems. The determination of when and how much to order is an inventory problem, but the difference is that the requirement function is controlled by the choice of replacement ages. On the other hand, the determimation of when to replace is an equipment replacement problem, but the decision is made under consideration of procurement and holding costs of spare units. A typical example of such a joint system of stocking and replacement is a maintenance department of a production plant or a factory, or a material supply depot in a military logistic system.

Several studies for such a problem have been done, but for the most part the inventory is limited to only a single spart unit[8, 10, 11, 12, 13, 18, 19]. Exceptionally, Falkner [2, 3] have treated joint stocking and replacement problems without limitation on the stocking level of spare units, but in his studies the procurement of spare units is allowed once only at the beginning of planning horizon.

In this paper we examine joint stocking and preventive age replacement policy without any restriction on stocking level and procurement time. Assuming an increasing failure rate(IFR) and instantaneous replenishment, we show that there exists a unique set of solutions for the joint stocking and preventive age replacement policy over an infinite time span, and present a procedure for determining order quantity per order and preventive replacement ages for the units.

List of Notations

The following notations are used throughout the paper:

$F(t)$: the probability density function of time to failure of a unit.

$f(t)$:the cumulative distribution function of time to failure of a unit.

$\bar{F}(t)$: $1-F(t)$, (the survivor function).

$h(t)$: $f(t)/\bar{F}(t)$, (failure rate at age t).

c_f: the expected cost of a corrective replacement; this includes the purcha sing price of replacing time.

c_p: the expected cost of a preventive replacement; this includes the purchasing price of replacing item($c_p < c_f$).

c_h: the inventory carrying cost per item per unit time.

c_o: the fixed ordering cost.

Q: the order quantity per order.

Q^*: the optimal value of Q.

$T_i(Q)$: the preventive replacement age for the i-th unit from the last[with (i-1) spares in inventory] for a given order quantity Q.

$T^*_i(Q)$: the optimal value of $T_i(Q)$ for a given order quantity Q.

$C(Q,\underline{T}(Q))$: the expected cost per unit time when order quantity is Q and sequence of preventive repla cement ages is $\underline{T}(Q)=\{T_i(Q)\} =\{T_Q(Q), T_{Q-1}(Q),\cdots,T_1(Q)\}$.

$C^*(Q)$: $C(Q, \underline{T^*}(Q))$, (the optimum cost per unit time for a given order quantity Q).

$C^*(Q^*)$: $\min C(Q, \underline{T^*}(Q))$, (the global optimum cost per unit time).

2. The Model

Under the joint stocking and preventive age replacement policy proposed in this paper, Q units are purchased per order, and the i-th unit from the last [with (i-1) spares] is replaced at failure or at age $T_i(Q)$, whchever occurs first. It is assumed that the replenishment is instantaneous and the failure rate is increasing. An age-based preventive replacement policy is meaningless for a decreasing failure rate. The time between successive orders is a cycle and the behavior in each cycle repeats. Thus, the expected cost per unit time for an infinite time span is the expected cost per cycle divided by the expected cycle length[15].

The expected cost per cycle is the sum of the ordering, replacement(corrective and preventive), and holding costs. The ordering cost per cycle is c_o. The expected cost of preventive and corrective replacement per cycle is

$$\Sigma_{i=1}^{Q} [c_f F(T_i(Q))+c_p\bar{F}(T_i(Q))]$$
$$=\Sigma_{i=1}^{Q} [c_p+(c_f-c_p)F(T_i(Q))].$$

Since the expected use time of the i-th unit from the last is

$$\int_0^{T_i(Q)} tf(t)dt+T_i(Q)\int_{T_i(Q)}^{\infty} f(t)dt$$
$$=\int_0^{T_i(Q)} \bar{F}(t)dt,$$

the expected cycle length is $\Sigma_{i=1}^{Q}\int_0^{T_i(Q)} \bar{F}(t)dt$, and the expected holding cost per cycle is $\Sigma_{i=1}^{Q} c_h(i-1) \int_0^{T_i(Q)} \bar{F}(t)dt$.

Hence, the expected cost per unit time is

$$C(Q, \underline{T}(Q)) = \frac{c_o + \Sigma_{i=1}^{Q}[c_p + (c_f - c_p)F(T_i(Q)) + c_h(i-1)\int_0^{T_i(Q)} \bar{F}(t)dt]}{\Sigma_{i=1}^{Q} \int_0^{T_i(Q)} \bar{F}(t)dt} \quad \cdots\cdots (1)$$

The problem is to select order quantity Q and preventive replacement ages $\underline{T}(Q) = \{T_Q(Q), T_{Q-1}(Q), \cdots, T_1(Q)\}$ so as to minimize $C(Q, \underline{T}(Q))$.

2.1 Determination of Preventive Replacement Ages

To find the optimal preventive replacement ages for a given Q, we set the partial derivatives of Equation(1) with respect to $T_i(Q)$ equal to zero, obtaining

$$(c_f - c_p)h(T^*_i(Q)) + c_h(i-1) = C(Q, \underline{T}^*(Q))$$
$$\text{for } i = 1, 2, \cdots, Q. \quad \cdots\cdots (2)$$

From Equation(2), we obtain the following recursive relationships among the optimal preventive replacement ages:

$$h(T^*_{i+1}(Q)) = h(T^*_i(Q)) - c_h/(c_f - c_p)$$
$$\text{for } i = 1, 2, \cdots, Q\text{-}1. \quad \cdots\cdots (3)$$

Notice that $T^*_{i+1}(Q) < T^*_i(Q)$ due to the inventory carrying cost c_h and thus the preventive replacement ages $\{T^*_Q(Q), T^*_{Q-1}(Q), \cdots, T^*_1(Q)\}$ form an increasing sequence.

Multiplying $\int_0^{T_i(Q)} \bar{F}(t)dt$ to both sides of Equation(2) and summing over all i, we obtain

$$\Sigma_{i=1}^{Q} \int_0^{T^*_i(Q)} \bar{F}(t)dt[(c_f - c_p)h(T^*_i(Q)) + c_h(i-1)]$$
$$= [\Sigma_{i=1}^{Q} \int_0^{T^*_i(Q)} \bar{F}(t)dt] C(Q, \underline{T}^*(Q))$$
$$= c_0 + c_p Q + \Sigma_{i=1}^{Q} (c_f - c_p)F(T^*_i(Q))$$
$$+ \Sigma_{i=1}^{Q} c_h(i-1) \int_0^{T^*_i(Q)} \bar{F}(t)dt$$

or

$$\Sigma_{i=1}^{Q} [h(T^*_i(Q)) \int_0^{T^*_i(Q)} \bar{F}(t)dt - F(T^*_i(Q))]$$
$$= (c_o + c_p Q)/(c_f - c_p). \quad \cdots\cdots (4)$$

From Equation(3), we obtain

$$h(T^*_2(Q)) = h(T^*_1(Q)) - c_h/(c_f - c_p),$$
$$h(T^*_3(Q)) = h(T^*_2(Q)) - c_h/(c_f - c_p),$$
$$\vdots$$
$$h(T^*_i(Q)) = h(T^*_{i-1}(Q)) - c_h/(c_f - c_p).$$

Adding these equations yields

$$h(T^*_i(Q)) = h(T^*_1(Q)) - (i-1)c_h/(c_f - c_p)$$

or

$$T^*_i(Q) = h^{-1}[h(T^*_1(Q)) - (i-1)c_h/(c_f - c_p)]. \quad \cdots\cdots (5)$$

Substituting Equation(5) into Equation(4), we can determine $T^*_1(Q)$. Once $T^*_1(Q)$ is obtained, we can determine $T^*_i(Q)$ using Equation(5). From Equation(2) the optimum cost is

$$C^*(Q) = (c_f - c_p)h(T^*_1(Q)). \quad \cdots\cdots (6)$$

Thus the optimal set of solutions satisfies Equations(4) and (5), and the corresponding cost is given by Equation(6).

Remark: If $h(\cdot)$ is strictly increasing, the left hand side of Equation(4) is also strictly increasing in $T_i^*(Q)$ starting from zero. Thus there exists a unique set of solutions $\underline{T}^*(Q)$(possibly infinite, that is, never to perform preventive replacement) satisfying Equations(4) and (5). Furthermore, the set of solutions $\underline{T}^*(Q)$ must yield $C^*(Q)$ for a given Q, since the Hessian matrix evaluated at the critical point is positive definite with positive diagonal elements $(c_f-c_p)[f'(T_i^*(Q))+h(T_i^*(Q))f(T_i^*(Q))]/\Sigma_{i=1}^{Q}\int_0^{T_i(Q)}\bar{F}(t)dt$. Notice that, since $h'(t)=[f'(t)+h(t)f(t)]/\bar{F}(t)$, IFR implies $[f'(t)+h(t)f(t)]>0$ for all t.

2.2 Determination of Order Quantity

The following theorem shows that $C^*(Q)$ is integer quasi-convex in Q[6].

Theorem 1. If h(t) is increasing in t, there exists $Q^*\geq 1$ such that

$$C^*(1)\geq C^*(2)\geq\cdots\geq C^*(Q^*)\leq C^*(Q^*+1)\leq\cdots.$$

Proof: See Appendix A1.

Hence, one for one ordering is never optimal if $C^*(1)>C^*(2)$. In order to find the condition that stocking is required, we now examine when $C^*(1)>C^*(2)$.

Theorem 2. If h(t) is strictly increasing in t, then $C^*(1)>C^*(2)$ is satisfied if and only if

$$h(T_o)\int_0^{T_o}\bar{F}(t)dt-F(T_o)>c_p/(c_f-c_p),$$

where $T_o=h^{-1}[h(T_1^*(1))-c_h/(c_f-c_p)]$.

Proof: See Appendix A2.

Thus if $T_1^*(1)$ (preventive replacement age for the Barlow and Proschan model[1]) is known, it can be decided wether stocking is required or not. If stocking is required, we can determine the optimum stocking level, since Q^* is the smallest positive integer Q which satisifies

$$C^*(Q+1)-C^*(Q)\geq 0$$

from Theorem 1.

3. Equal Preventive Replacement Age Case

If we assume that preventive replacement ages are equal, the problem is simplified. Suppose that $T_i(Q)=T(Q)$ for all i. Then Equations(1) and (4) degenerate to

$$C(Q,T(Q))=\frac{c_o+c_pQ+(c_f-c_p)QF(T(Q))+c_h\,Q(Q-1)\int_0^{T(Q)}\bar{F}(t)dt/2}{Q\int_0^{T(Q)}\bar{F}(t)dt} \quad\cdots\cdots\cdots\cdots (7)$$

and

$$h(T^*(Q))\int_0^{T^*(Q)}\bar{F}(t)dt-F(T^*(Q))=(c_o/Q+c_p)/(c_f-c_p). \quad\cdots\cdots\cdots\cdots (8)$$

Substituting the $T^*(Q)$ satisfying Equation(8) into Equation(7), we obtain

$$C^*(Q)=(c_f-c_p)h(T^*(Q))+c_h(Q-1)/2. \quad (9)$$

The result coincides with the ordinary inventory system of Sivazlian[17], and the optimal order quantity Q^* is approximately equal to Wilson's lot size formula $\sqrt{2\lambda c_0/c_h}$.

4. Exponential Lifetime Case

Let us consider the case of exponential lifetime distribution, in which the failure rate is a constant.

Lemma. If $h(t)$ is a constant λ, $T^*_i(Q)$ is either 0 or ∞ for all $i=1,2,\cdots,Q$.

Proof: See Appendix A3.

From the above Lemma, we have the following theorem.

Theorem 3. If $h(t)$ is a constant λ, $T^*_i(Q^*)$ is ∞ for all $i=1,2,\cdots,Q^*$.

Proof: The proof is trivial from the Lemma, since $T^*_i(Q^*)=0$ means that the unit is condemned without any use and thus $C^*(Q^*)\geq C^*(Q^*-1)$ (This contradicts that Q^* is the optimal order quantity).

Thus, in the case of exponential lifetime, preventive replacement is unnecessary and each unit should be replaced only at failure.

Corollary. If $h(t)$ is a constant λ, then the optimal order quantity Q^* is the smallest positive integer Q which satisfies $Q(Q+1)\geq 2\lambda c_0/c_h$.

Proof: From Equation(1), $C(Q, \{\infty\})=\lambda c_0/Q+\lambda c_f+c_h(Q-1)/2$. Since $C(Q, \{\infty\})$ is convex in Q, Q^* is the smallest positive integer Q which satisfies $C(Q+1, \{\infty\})-C(Q, \{\infty\})\geq 0$ or $Q(Q+1)\geq 2\lambda c_0/c_h$.

5. Numerical Example and Comparative Cost Behavior

Consider a unit having a Weibull lifetime distribution $F(t)=1-\exp(-2t^2)$. Suppose that $c_0=\$10$, $c_f=\$50$, $c_p=\$10$ and $c_h=\$8$. From Equation(5), $T^*_i(Q)=T^*_1(Q)-(i-1)/20$. Substituting $T^*_i(Q)$ into Equation(4), we determine $T^*_i(Q)$ numerically and, from Equation(6), we obtain $C^*(1)=\$83.75$, $C^*(2)=\$75.12$, $C^*(3)=\$75.04$ and $C^*(4)=\$76.25$. Since $C^*(2)>C^*(3)$ and $C^*(3)<C^*(4)$, the optimal order quantity $Q^*=3$. The optimal preventive replacement ages are $T^*_3(3)=0.369$, $T^*_2(3)=0.419$, $T^*_1(3)=0.469$ and the optimum cost is $C^*(3)=\$75.04$.

Likewise, in equal preventive replacement age case from Equations(8) and (9), $Q^*=3$, $T^*(3)=0.420$ and the optimum cost is \$75.20.

Since the optimal order quantity Q^* is determined by a trade-off between the fixed ordering cost c_0 and the inventory carrying cost c_h, the expected cost per unit time is computed numerically and plotted as a continuous function of c_0/c_h as c_0 varies. Fig. 1 shows the respective costs of

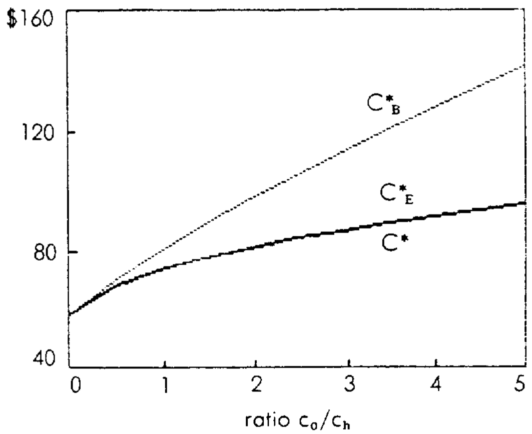

Fig. 1 Comparative cost behavior.

the three models (C^*_B: Barlow and Proschan's, C^*_E: equal preventive replacement age case, C^*: unequal preventive replacement age case). As might be expected, if $c_o=0$ the costs of the three models are equal. As the ratio c_o/c_h increases, cost reduction due to quantity purchase, C^*_B-C^*_E or C^*_B-C^*, increases. However, the difference between C^*_E and C^* is very little regardless of the ratio c_o/c_h.

In order to see the relative inefficiency of equal replacement ages over unequal replacement ages, 16 example problems are solved and summarized in Table 1 for all the combinations of parameter values half and twice as much as the original para-

Table 1. Comparative cost (Unequal replacement ages vs. equal replacement ages)

No.	Cost data ($)				Expected cost ($)		Relative inefficiency
	c_o	c_f	c_p	c_h	C_E^*	C^*	$(C_E^*-C^*)/C^*$
1	5	25	5	4	37.60	37.52	0.0021
2	20	25	5	4	47.45	47.19	0.0055
3	5	100	5	4	75.25	75.22	0.0005
4	5	25	20	4	45.90	45.88	0.0004
5	5	25	5	16	41.88	41.88	0.0000
6	20	100	5	4	89.70	89.06	0.0072
7	20	25	20	4	53.87	53.86	0.0002
8	20	25	5	16	60.27	60.00	0.0045
9	5	100	20	4	125.00	124.63	0.0030
10	5	100	5	16	83.70	83.13	0.0069
11	5	25	20	16	47.8[illegible]	47.81	0.0000
12	20	100	20	4	135.19	135.00	0.0014
13	20	100	5	16	111.00	109.84	0.0106
14	20	25	20	16	63.86	63.83	0.0005
15	5	100	20	16	130.00	130.00	0.0000
16	20	100	20	16	150.40	150.08	0.0021

meters. The results show that the maximum relative inefficiency is about 1%. It may be satisfactory to use equal preventive replacement age model in practice.

6. Concluding Remarks: Optimal Stocking with Lead Time

In this paper, a procedure is presented for determining order quantity and preventive replacement ages in a joint stocking and replacement problem. Since the formulation assumes that the replenishment is instantaneous, a reorder is placed as soon as a stockout occurs, and idletime(no unit in use) does not exists.

If the procurement lead time is not negligible, we must determine the reorder point as well as order quantity and preventive replacement ages. In this case, since the cycle length depends on the age of the unit in use when delivery takes place, the problem is not tractable for a general lifetime distribution.

In the case of exponential lifetime, however, the problem simplifies to that of determining order quantity and reorder point, since preventive replacement is unnecessary. The "memoryless" property of the exponential distribution enables us to start the process anew when delivery takes place, regardless of the age of the unit in use.

Consider the case with exponential lifetime and lead time distributions with means $1/\lambda$ and $1/\mu$, respectively. Then, the expected cost per unit time is, from Appendix A4,

$$C(Q, r)= \frac{c_0+c_hQ[(Q-1)/2\lambda+r/\lambda-1/\mu+\lambda^r/\mu(\lambda+\mu)^r]+c_s\lambda^r/\mu(\lambda+\mu)^r}{Q/\lambda+\lambda^r/\mu(\lambda+\mu)^r} \quad \cdots\cdots (10)$$

where r is the reorder point and c_s is the idleness cost per unit time. Since having the unit inoperative is a significant cost factor, it might be assumed that the length of stockout period($\lambda^r/\mu(\lambda+\mu)^r$) is small enough to be neglected compared to the length of the operating period(Q/λ) as in Hadley and Whitin[4]. Then, the expected cost in Equation(10) approximates to

$$\hat{C}(Q, r)=\lambda c_0/Q+c_h\lambda[(Q-1)/2\lambda+r/\lambda-1/\mu+\lambda^r/\mu(\lambda+\mu)^r]+c_s\lambda^{r+1}/Q\mu(\lambda+\mu)^r. \quad \cdots\cdots (11)$$

To find the optimal pair (Q^*, r^*), an iterative procedure as in Hadley and Whitin[4] can be used. Since $\hat{C}(Q, r)$ is convex, the solution(Q^*, r^*) obtained from the iterative procedure yields an absolute minimum.

References

[1] Barlow, R.E. and Proschan, F., *Mathematical Theory of Reliability,* John Wiley & Sons, New York, pp.85-95(1965)

[2] Falkner, C.H., "Jointly Optimal Inventory and Maintenance Policies for Stochastically Failing Equipment", *Ope-*

rations Research, **16**, 3, pp.587-601(1968)

[3] Falkner, C.H., "Optimal Spares for Stochastically Failing Equipment", *Naval Research Logistics Quarterly,* **16**, 3, 287-295(1965)

[4] Hadly, G. and Whitin, T.M., *Analysis of Inventory Systems,* Prentice-Hall, Inc., Englewood Cliffs, New Jersey, pp.168-175(1963)

[5] Karlin, S., "The Application of Renewal Theory to the Study of Inventory Policies", Chapter 15 in *Studies in the Mathematical Theory of Inventory and Production,* K.J. Arrow, S. Karlin, H. Scarf, Stanford University Press, Stanford, California(1958)

[6] Kumin, H., "On Characterizing the Extrema of a Function of Two Variables, One of Which Is Discrete", *Management Science,* **20**, 2, pp.126-129(1973)

[7] McCall, J.J., "Maintenance Policies for Stochastically Failing Equipment: A Survey", *Management Science,* **11**, 5, pp.493-524(1965)

[8] Mehrez, A. and Stulman, A., "Age Replacement in the Presence of Inventory Constraints", *European Journal of Operational Research,* **12**, 2, pp.183-189(1983)

[9] Nguyen, D.G. and Murthy, D.N.P., "Optimal Preventive Maintenance Policies for Repairable Systems", *Operations Research,* **29**, 6, pp.1181-1194(1981)

[10] Osaki, S., "An Ordering Policy with Random Lead Time", *International Journal of Systems Science,* **8**, 10, pp.1091-1095(1977)

[11] Osaki, S., Kaio, N., and Yamada, S., "A Summary of Optimal Ordering Policies", *IEEE Transactions on Reliability,* **R-30**, 3, pp.272-277(1981).

[12] Park, K.S. and Park, Y.T., "Ordering Policies under Minimal Repair", *IEEE Transactions on Reliability,* **R-35**, 1, pp.82-84(1986)

[13] Park, Y.T. and Park, K.S., "Generalized Spare Ordering Policies with Random Lead Time", *European Journal of Operational Research,* **23**, 3, pp.320-330(1986)

[14] Pierskalla, W.P. and Voelker, J.A., "A Survey of Maintenance Models: The Control and Surveillance of Deteriorating Systems", *Naval Research Logistics Quarterly,* **23**, 3, pp.353-388(1976)

[15] Ross, S.M., *Applied Probability Models with Optimization Applications,* Holden-Day, San Francisco, pp.51-54(1970)

[16] Sherif, Y.S. and Smith, M.L., "Optimal Maintenance Models for Systems Subject to Failure-A Review", *Naval Research Logistics Quarterly,* **28**, 1, pp.47-74(1981)

[17] Sivazlian, B.D., "A Continuous Review(s, S) Inventory System with Arbitrary Interarrival Distribution between Unit Demand", *Operations Research,* **22**, 1, pp.65-71(1974)

[18] Thomas, L.C. and Osaki, S., "An Optimal Ordering Policy for a Spare Unit with Lead Time", *European Journal of Operational Research,* **2**, 6, pp.409-419(1978)

[19] Wiggins, A.D., "A Minimum Cost

Model of Spare Parts Inventory Control", *Technometrics,* 9, 4, pp.661-665(1967)

Appendices

A1. Proof of Theorem 1

Since the optimal set of solutions satisfies Equations(4) and (5), $T^*_1(Q)$ is the solution of

$$\Sigma_{i=1}^{Q} [h(T_i) \int_0^{T_i} \bar{F}(t)dt - F(T_i)] = (c_o + c_p Q)/(c_f - c_p), \quad \cdots\cdots\cdots (A1)$$

where all $T_i(; 1<i\le Q)$ are related to T_1 by

$$T_i = h^{-1}[h(T_1) - (i-1)c_h/(c_f - c_p)]. \quad \cdots\cdots (A2)$$

Denote the left hand side and the right hand side of Equation(A1) by $g(Q, T_1)$ and $f(Q)$, respectively. Define

$$\triangle g(Q, T_1) = g(Q+1, T_1) - g(Q, T_1)$$

and

$$\triangle f(Q) = f(Q+1) - f(Q).$$

We first show that $g(Q, T_1)$ and $f(Q)$ satisfy the following three conditions in the Lemma 2 of Nguyen and Murthy[9]:

C1: $g(Q, T_1)$ is increasing in T_1,
C2: $\triangle g(Q, T_1)$ is decreasing in Q,
C3: $\triangle f(Q)$ is a constant.

Condition C1 is shown to be satisfied by noting that as T_1 increases, $T_i(;1<i\le Q)$ also increase and hence $g(Q, T_1)$ is increasing in T_1. Condition C2 is satisfied, since

$$\triangle g(Q, T_1) = h(T_{Q+1}) \int_0^{T_{Q+1}} \bar{F}(t)dt - F(T_{Q+1})$$

and

$$T_{i+1} \le T_i \text{ for all } i,$$

which imply that as Q increases, T_{Q+1} decreases and thus $\triangle g(Q, T_1)$ decreases. Condition C3 is also satisfied, since $\triangle f(Q) = c_p/(c_f - c_p)$.

Hence from the Lemma 2 of Nguyen and Murthy[9], there exists $Q^* \ge 1$ such that

$$T^*_1(1) \ge T^*_1(2) \ge \cdots \ge T_1(Q^*-1) \ge T^*_1(Q^*) \le T^*_1(Q^*+1) \le \cdots,$$

which implies, from Equation(6),

$$C^*(1) \ge C^*(2) \ge \cdots \ge C^*(Q^*-1) \ge C^*(Q^*) \le C^*(Q^*+1) \le \cdots.$$

A2. Proof of Theorem 2

Suppose that $C^*(1) > C^*(2)$. Since $h(t)$ is strictly increasing in t, $h(t) \int_0^t \bar{F}(x)dx - F(t)$ is also strictly increasing in t. From Equation(4),

$$c_p/(c_f - c_p) = \Sigma_{i=1}^{2} [h(T^*_i(2)) \int_0^{T^*_i(2)} \bar{F}(t)dt - F(T^*_i(2))] - [h(T^*_1(1)) \int_0^{T^*_1(1)} \bar{F}(t)dt - F(T^*_1(1))].$$

Since $C^*(1) > C^*(2)$ implies, from Equation(6), $T^*_1(1) > T^*_1(2)$,

$$c_p/(c_f - c_p) < \Sigma_{i=1}^{2} [h(T^*_i(2)) \int_0^{T^*_i(2)} \bar{F}(t)dt - F(T^*_i(2))]$$

$$-[h(T^*_1(2)) \int_0^{T^*_1(2)} \bar{F}(t)dt - F(T^*_1(2))]$$

or

$$c_p/(c_f - c_p) < h(T^*_2(2)) \int_0^{T^*_1(2)} \bar{F}(t)dt - F(T^*_2(2)),$$

where $T^*_2(2) = h^{-1}[h(T^*_1(2)) - c_h/(c_f - c_p)]$ from Equation(5).

Since T_o is defined as $h^{-1}[h(T^*_1(1)) - c_h/(c_f - c_p)]$ and $T^*_1(1) > T^*_1(2)$, $T_o > T^*_2(2)$ and thus

$$c_p/(c_f - c_p) < h(T_o) \int_0^{T_o} \bar{F}(t)dt - F(T_o).$$

Similarly, we can prove the contraposition of the sufficiency: if $C^*(1) \leq C^*(2)$, then the opposite is true.

A3. Proof of Lemma

Since $f(t) = \lambda \bar{F}(t)$ and $\int_0^t \bar{F}(x)dx = F(t)/\lambda$, differentiating Equation(1) with respect to $T_i(Q)$ yields

$$\partial C(Q, \underline{T}(Q))/\partial T_i(Q)$$
$$= \lambda \bar{F}(T_i(Q)) [-\lambda (c_o + c_p Q) + c_h \Sigma_{j=1}^{i-1} (i-j) F(T_j(Q)) - c_h \Sigma_{j=i+1}^{Q} (j-i) F(T_j(Q))] / [\Sigma_{j=1}^{Q} F(T_j(Q))]^2$$

Since the sign of $\partial C(Q, \underline{T}(Q))/\partial T_i(Q)$ is independent of $T_i(Q)$, $T^*_i(Q) = 0$ when the derivative is positive, and $T_i^*(Q) = \infty$ when the derivative is negative. If the derivative is zero, all values of $T_i(Q)$ give the same cost, and both 0 and ∞ are as good as any. Thus, and optimal $T^*_i(Q)$ is either 0 or ∞.

A4. Derivation of C(Q,r) in Eqation(10)

Suppose that the lead time is a constant α. From the equations(60) and (62) of Karlin [5], after some notational changes ($T \to Q$, $R \to r$, $h \to c_h$), the expected holding cost per cycle is

$$K_h(Q, r, \alpha) = c_h Q[(Q-1)/2\lambda + (r/\lambda) \Sigma_{k=0}^{r-1} \varphi(k) - \alpha \Sigma_{k=0}^{r-2} \varphi(k)],$$

and the expected cycle length is

$$T(Q, r, \alpha) = \alpha + Q/\lambda - (r/\lambda) \Sigma_{k=r}^{\infty} \varphi(k) - \alpha \Sigma_{k=0}^{r-2} \varphi(k),$$

where

$$\varphi(k) = (\lambda\alpha)^k \exp(-\lambda\alpha)/k!.$$

Karlin[5] also presented the relevant probability quantities for computing C(Q, r) when the lead time is a random variable with density $g(x) = \mu \exp(-\mu x)$, but his results contain some errors. Correct derivation follows.

If the lead time distribution $g(x) = \mu \exp(-\mu x)$, the expected holding cost per cycle is

$$K_h(Q,r) = \int_0^{\infty} K_h(Q, r, x) g(x)dx$$
$$= c_h Q[(Q-1)/2\lambda + (r\mu/\lambda^2) \Sigma_{k=0}^{r-1} \{\lambda/(\lambda+\mu)\}^{k+1} - (\mu/\lambda^2) \Sigma_{k=0}^{r-2} (k+1)\{\lambda/(\lambda+\mu)\}^{k+2}]$$
$$= c_h Q[(Q-1)/2\lambda + r/\lambda - 1/\mu + \lambda^r/\mu(\lambda+\mu)^r],$$

and the expected cycle length is

$$
\begin{aligned}
T(Q, r) &= \int_0^\infty T(Q, r, x)\, g(x)dx \\
&= 1/\mu + Q/\lambda - (r\mu/\lambda^2) \\
&\quad \Sigma_{k=r}^{\infty} \{\lambda/(\lambda+\mu)\}^{k+1} - (\mu/\lambda^2) \\
&\quad \Sigma_{k=0}^{r-2} (k+1)\{\lambda/(\lambda+\mu)\}^{k+2} \\
&= Q/\lambda + \lambda^r/\mu(\lambda+\mu)^r.
\end{aligned}
$$

Since the expected idletime per cycle is $\lambda^r/\mu(\lambda+\mu)^r$[5], the expected total cost per cycle is

$$
K(Q, r) = c_o + c_h Q[(Q-1)/2\lambda + r/\lambda - 1/\mu + \lambda^r/\mu(\lambda+\mu)^r] + c_s \lambda^r/\mu(\lambda+\mu)^r.
$$

$$
C(Q,r) = \frac{K(Q, r)}{T(Q, r)} = \frac{c_o + c_h Q[(Q-1)/2\lambda + r/\lambda - 1/\mu + \lambda^r/\mu(\lambda+\mu)^r] + c_s \lambda^r/\mu(\lambda+\mu)^r}{Q/\lambda + \lambda^r/\mu(\lambda+\mu)^r}
$$