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# A Stochastic Production Planning Problem in Rolling Horizon Environment

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# 계획기간의 연동적 고려 경우의 추계적 생산계획

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## Abstract

This paper considers a single-product production and inventory management problem where cumulative demands up to each time period are mutually independent random variables(known) having continuous probability distributions and the associated cost-minimizing production schedule(when to produce and how much to produce) need be determined in rolling horizon environment. For the problem, both the production cost and the inventory holding and backlogging costs are included in the whole system cost. The probability distributions of these costs are expressed in terms of random demands, and utilized to exploit a solution procedure for a production schedule which minimizes the expected unit time system cost and also reduces the probability of risk that, for the first-period of each production cycle(rolling horizon), the cost of the "production" option will exceed that of the "non-production" one. Numerical examples are presented for the solution procedure illustration.

## 1. Introduction

This paper analyzes a single-product production and inventory management prolem where cumulative demands up to each time period (partial sums of demands) are mutually independent random variables(known) having continuous probability distributions, and where capacity constraints are imposed on each produc-

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tion. Backlogging is also allowed. Furthermore, both the production cost and the inventory holding and backlogging costs are included in the whole system cost. The production cost is composed of a setup cost for each production setup and of a linear cost proportional to production quantity, and the inventory holding and backlogging costs are represented by costs linearly proportional to inventories on hand and backlogged, respectively. For the problem, it is assumed that production schedules are reviewed at each period and that the lead time for each replenishment is zero. With these problem specifications, a cost-minimizing production schedule shall be determined in rolling horizon environment.

For problems with deterministic time-dependent demands, various solution procedures for the first-period production decision in ech rolling horizon have been proposed in literature. For example, Silver and Meal[7] have proposed a procedure(known as cost effective) for selecting production quantities to minimize the cost per unit time (see also Baker[2], Sung and Park[8]). However, Wemmerlov and Whybark[9] have shown that the solution procedures appeared in literature for problems with deterministic demands may not maintain their effectiveness when applied to those with random demands(see also De Bodt and Van Wassenhove[5]).

For problems with random demands, a few of research results have been reported in literature. Silver[6] has suggested a three-stage procedure of determining whether or not to produce at the first-period of each rolling horizon, and of specifying both the number of demand periods to cover and the production quantity if the first-period decision takes the production option. However the procedure uses the expected values of demands rather than demand distributions themselves, so that it only incorporates the expected demand data into a deterministic model.

Askin[1] has additionally incoporated the cost effects of probabilistic demands to the problem of Silver[6]. Bookbinder and H'ng [4] have investigated a similar problem for which the solution procedure was designed for deterministic values transformed from probabilistic demands in terms of customer service level. Barnes, Zinn, and Elderd [3] have analyzed a probabilistic cash flow problem to character-ize the probability of the present values of profiles and to determine an optimal profile which stochastically dominates other profiles.

The objective of this paper shall now be stated. It is required under the rolling horizon environment that one of the two options between "production and non-production" be decided for the first-period of each production cycle(rolling horizon), where a production cycle is the time interval over which all the demands are satisfied by the first-period production. If enough inventory is available at the first period, then the non-production option may be preferable. Otherwise, the produciton option may be selected. It may rather be more rigorous to state such option preferences in probability measure, since there may exist a positive probability that the production(non-production) option may cost more than the non-production(production) one.

Thus, the objective of this paper is to find a production schedule in rolling horizon environment which minimizes the expected unit time cost, and also reduces the probability of risk that the production option may cost more than the non-production one. In other words, the production schedule is determined upon a decision strategy mixture of the risk probability and the expected unit time system cost. The motivation of proposing the decision strategy mixture can further be described.

If a decision is made to setup a production at the first peroid, then the decision may be interpreted as representing the situation where the expected cost incurred by taking the non-production option is figured out to be more expensive than the expected unit time cost incurred by taking the production option. By the way, due to the probabilistic nature of the system cost, there may still exist a risk probability that the random cost incurred by production in the first period may exceed that of non production. Moreover, the production quantity selected only on the basis of the expected unit time cost may not be consistent with that determined based on the risk probability.

## 2. Analysis

For the problem analysis, following notations are introduced.

$T$ = rolling(forecast) horizon.

$C$ = production capacity at the first period of each production cycle.

$D_t$ = non-negative random cumulative demand for the time interval from period 1(the first period) through period $t(t=1, 2, \cdots, T)$.

$f_t(D_t)$ = pdf of $D_t$, $(t=1, 2, \cdots, T)$.

$F_t(D_t)$ = cdf of $D_t$, $(t=1, 2, \cdots, T)$.

$p$ = unit production cost at period 1.

$A$ = production setup cost at period 1.

$h_t$ = inventory holding cost per unit on hand at period $t,(t=1, 2, \cdots, T)$.

$b_t$ = inventory backlogging cost per unit backlogged at period $t,(t=1, 2, \cdots, T)$.

$w$ = inventory position at period 1.

$Q$ = production quantity at period 1.

$R=w+Q$ = production level as production is setup.

Note that period t representes th $t^{th}$ period from the first one (period 1) in each production cycle.

Probability distributions of the unit time system cost shall now be derived. Let $K(R, t)$ be the total system cost incurred over the periods from period 1 through period t where production is setup at the first period and the production level R amounts to all the demands required during the interval [1, t ], (t=1, 2, ⋯, T), and $K_w$ be the total system cost incurred during the interval [1, t] where production is not setup at the first period but all the demands are taken care of by inventory. The respective mathematical expressions are given as follow :

$$K(R, t)=A+p(R-w)+\Sigma_{i=1}^{t}[h_i\max\{0, R-D_i\} + b_i\max\{0, D_i-R\}]$$

and

$$K_W=h_1\max\{0, w-D_1\}+b_1\max\{0, D_1-w\}.$$

Consider the variable $X_i$

$$X_i=h_i\max\{0, R-D_i\}+b_i\max\{0, D_i-R\}.$$

Then $X'_i$s are mutually independent, since $D'_i$s are mutually independent.

Therefore,

$$P\{X_i\le x\}=P\{R-\frac{x}{h_i}\le D_i\le R+\frac{x}{b_i}\}$$
$$=F_i(R+\frac{x}{b_i})-F_i(R-\frac{x}{h_i}),\ \text{for } x\ge 0,$$

and

$$\alpha_i(x, R)=\begin{cases} f_i(R+\frac{x}{b_i})/b_i+f_i(R-\frac{x}{h_i})/h_i, & 0\le x\le h_iR, \\ f_i(R+\frac{x}{b_i})/b_i, & x>h_iR, \end{cases}$$

where $\alpha_i(x, R)$ represents the pdf of $X_i$.

Denoting by $\beta_t(x, R)$ and $\psi_t(x, R)$ the pdf and cdf of $Y_t=\Sigma_{i=1}^{t}X_i$, (t=1, 2, ⋯, T), respectively, both $\beta_t(x, R)$ and $\psi_t(x, R)$ can be determined from $\alpha_i(x, R)$ by use of the Laplace transform theory. Thus, the probability of the unit time system cost incurred due to the production set at the level R to satisfy all the demands over the production cycle[1, t] can be expressed as follow:

$$P\{K(R, t)/t\le x\}$$
$$=P[A+p(R-w)+\Sigma_{i=1}^{t}\{h_i\max\{0, R-D_i\} + b_i\max\{0, D_i-R\}\}\le tx]$$
$$=\begin{cases} \psi_t(tx-A-p(R-w), R), & \text{if } x\ge\max\{0, \frac{A+p(R\text{-}w)}{t}\}, \\ 0, & \text{otherwise,} \end{cases}$$

and, for $x\ge\max\{0, \frac{A+p(R\text{-}w)}{t}\}$,

$$G_t(x, R)=\psi_t(tx-A-p(R-w), R)$$
$$g_t(x, R)=t\cdot\beta_t(tx-A-p(R-w), R),$$

where $G_t(x, R)$ and $g_t(x, R)$ are the cdf and the pdf of K(R, t)/t, respectively.

The above relations imply that the pdf and cdf of the unit time cost K(R, t)/t can be derived from the pdf's of demands $D_1$, $D_2$, ⋯, $D_t$(t=1, 2, ⋯, T). Similary, the pdf and cdf of the non-production cost $K_w$ can be derived from the pdf of $D_1$. In other words, the probability of the unit time system cost incurred due to the non-production option at the first period is given.

$$P\{K_w\le x\}=P[h_1\max\{0, w-D_1\}+b_1\max\{0, D_1-w\}\le x]=P\{w-\frac{x}{h_1}\le D_1\le w+\frac{x}{b_1}\}$$
$$=\begin{cases} F_1(w+\frac{x}{b_1})-F_1(w-\frac{x}{h_1}), & x\ge 0 \\ 0, & \text{otherwise} \end{cases}$$

$$G_w(x)=F_1(w+\frac{x}{b_1})-F_1(w-\frac{x}{h_1}),\ x\geq 0$$

$$g_w(x)=\begin{cases} f_1(w+\frac{x}{b_1})/b_1+f_1(w-\frac{x}{h_1})/\ h_1, & 0\leq x\leq h_1 w \\ f_1(w+\frac{x}{b_1})/b_1, & x>h_1 w, \end{cases}$$

where $G_w(x)$ and $g_{(}x)$ are the cdf and the pdf of $K_w$, respectively.

Now, teh expected unit time system cost can be derived and then used to find the optimal production level for a production cycle [1, t](t=1, 2, ⋯, T)

$$E[K(R, t)]=A+p(R-w)+\Sigma_{i=1}^{t}b_i(E[D_i]-R)$$
$$+\Sigma_{i=1}^{t}(h_i+b_i)\int_0^R(R-x)f_i(x)dx$$
$$E[K_w]=b_1(E[D_1]-w)+(h_1+b_1)\int_0^w(w-x)\ f_1(x)dx$$

Let $R_t^*$ represent the optimal production level at a given period t (t=1, 2, ⋯, T), such that

$$R_t^*=\arg\min\{E[K(R, t)/t]\mid w<R_t\leq C+w\}.$$

In fact,$R_t^*$ values can be obtained from E [K(R, t)] by solving the following equation:

$$\Sigma_{i=1}^{t}(h_i+b_i)F_i(R_t)=\Sigma_{i=1}^{t}b_i-p, \text{ for all } t.$$

Letting $\bar{R}_t$ satisfy the above equation, it follows that, since E[K(R, t)/t] is a convex function of R,

$$R_t^*=\begin{cases}\bar{R}_t, \text{ if } w<\bar{R}_t\leq C+w \\ C+w, \text{ if } \bar{R}_t>C+w.\end{cases}$$

It is noticed that if $\bar{R}_t$ is not greater than the inventory position w, then the corresponding number of periods to cover (which is t) is not justified. It would rather represent the situation that the corresponding expected unit time cost due to the production option is definitely greater than the cost due to the non-production option.

## 3. Risk-Averse Solution Procedure

Consider a solution procedure where the first-period production quantity decision is made so as to minimize the probability of risk that the expected unit time system cost associated with the first-period production quantity may exceed the expected cost incurred due to the non-production option. The solution procedure will be referred to, for the rest of the paper, as a "risk-averse" solution procedure.

Let two production levels $R^1$ and $R^2$ satisfy the cumulative demands for the first t and s($t\neq s$) periods, respectively. If it holds that $P\{K(R^1, t)/t>x\}\leq P\{K(R^2, s)/s>x\}$, for all $x\geq 0$, then it is said that $R^1$ stochastically dominates $R^2$. This is interpreted as $R^1$ is more risk-averse than $R^2$ in global sense. Therefore, $R^1$ is preferable. If it holds that $P\{K(R^1,t)/t>E[K_W]\}\leq P\{K(R^2,s)/s>E[K_W]\}$, then $R^1$ is said to be more risk-averse than $R^2$ in local sense at the expected cost of non-production(in the first period of each production cycle).

It may not be easy to find such a risk-averse solution in global sense, since the probability distribution of the system cost varies with the decision variable R itself. Thus, the local-sense risk-averse solution at the expected cost of non-production will be used here. On this understanding, the decision strategy mixture measured in both the probability of risk and the expected unit time system cost shall now be specified.

Let Ω be the set of the candidate number of periods to cover, so that

$$\Omega=\{t \mid E[K(R^*_t, t)/t]<E[K_w],\ \forall t, \text{ such that } \bar{R}_t>w\}.$$

Then, the decision on whether to take the production option or the non-production option for the first period is made as follows :

a) If $\Omega=\{\phi\}$, then take the non-production option.

b) If $\Omega\neq\{\phi\}$, then setup a production at the quantity $R^*_s$-w, where $R^*_s$ is determined at the level satisfying the relation

$$P\{K(R^*_s, s)/s>E[K_w]\}=\min\{P[K(R^*_t, t)/t>E[K_w] \mid t\in\Omega\}.$$

This decision structure indicates that the candidate number of periods to cover and the corresponding production quantity are determined by considering sequentially the expected unit time system cost and the probability of risk as follow

a) Find all the candidate periods t such that each corresponding expected unit time system cost of the production option is preferable over that of the non-production one.

b) Then, select the best period t where the associated $R^*_t(t\in\Omega)$ gives the minimum probability of risk.

## 4. Numerical Example

An illustrative example is stated as follows :

a) The planning horizon is over 5 periods.

b) The random demands at each of the 5 periods have exponential distributions with means 110, 40, 10, 62, and 12 units, respectively.

c) The production capacity is set at 300 units in each period.

d) All the cost data are given in Table 1.

Two initial inventory positions (w=0, w=98) are considered, and their associated solutions are shown in Tables 2 and 3.

Table 1. Problem Data

| Period t | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| mean demand | 110 | 40 | 10 | 62 | 12 |
| $E[D_t]$ | 110 | 150 | 160 | 222 | 234 |

| | |
|---|---|
| A=$48, | p=$1/unit, |
| h=$0.5/unit/period, | p=$12/unit/period |

Table 2. Summary of Solution with w=0.

| $t$ | $\bar{R}_t$ | $R^*_t$ | $E[K(R^*_t, t)/t]$ | $P\{k(R^*_t, t)/t > E[K_w]\}$ |
|---|---|---|---|---|
| 1 | 223.23 | 223.23 | 508.55 | 0.07 |
| 2 | 330.50 | 330.50 | 427.11 | 0.07 |
| 3 | 382.90 | 382.90 | 388.76 | 0.06 |
| 4 | 463.09 | 398.00 | 429.55 | 0.07 |
| 5 | 522.34 | 398.00 | 466.81 | 0.09 |
| $E[K_w]=1215.50$, $\Omega=\{1, 2, 3, 4, 5\}$, $R^*_s=R^*_3=382.90$ | | | | |

Table 3. Summary of Solution with w=98.

| $t$ | $\bar{R}_t$ | $R^*_t$ | $E[K(R^*_t, t)/t]$ | $P\{k(R^*_t, t)/t > E[K_w]\}$ |
|---|---|---|---|---|
| 1 | 223.00 | 223.23 | 410.55 | 0.10 |
| 2 | 330.50 | 330.50 | 378.11 | 0.11 |
| 3 | 382.90 | 382.90 | 356.09 | 0.17 |
| 4 | 463.09 | 398.00 | 405.05 | 0.27 |
| 5 | 522.34 | 398.00 | 447.21 | 0.27 |
| $E[K_w]=531.98$, $\Omega=\{1, 2, 3, 4, 5\}$, $R^*_s=R^*_1=223.23$ | | | | |

In Table 2, $R^*_3$ minimizes both the expected unit time cost and the probability of risk, while, in Table 3, it does not minimize the probability of risk but the expected unit time cost. This variation is due to the initial inventory which does not effect $R^*_t$ values but the system costs.

## 5. Concluding Remarks

This paper has considered, in rolling horizon environment, a single -product production planning model with random demands and exploited a soultion procedure using two measures such as expected unit time system cost and probability of risk. The incorporation of the risk probability measure may get the model practically meaningful in the real world where random demand occurrences are common. It follows that the model may be preferable over any other reference work incorporating only a single value information (e.q., only an expected demand).

In the solution procedure, the two measures are just proposed to use individually for a solution search. However, a product measure of the two measure outcomes may also be tried as an alternative.

For immediate applications of the model in addition to prodcution plannings in manufacturing industry, periodwise financing problems and maintenance service planning problems can be suggested. The model may be extended to the cases of problems with partial backlogging allowed or problems with risk probability measured in specific service levels.

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