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CHARACTERIZATION OF A GAUSSIAN COVARIANCE

HI JA SONG

1. Introduction

In 1977 S. Chobanjan and V. Tarieladze [1] showed that the necessary and sufficient condition for a nonnegative symmetric bounded linear operator $\boldsymbol{R}$ from X^* into X, where X is a Banach space which has cotype p for some $p<\infty$ and an unconditional basis $\{x_k\}_{k=1}^{\infty}$, to be a covariance operator of a Gaussian measure on X is that the series $\sum_{k=1}^{\infty} <\boldsymbol{R}x_k^*, x_k^*>^{\frac{1}{2}} x_k$ is convergent in X, where $\{x_k^*\}$ is the sequence of biorthogonal functionals associated with the basis $\{x_k\}$.

In 1978 S. Chobanjan, W. Linde and V. Tarieladze [2] extended the above result as follows: (a) Let X be a Banach space which contains no subspace isomorphic to C_0. A bounded linear operator T from a Hilbert space H into X is γ-Radonifying if and only if T is γ-summing. (b) Let X be a Banach space of type 2. A nonnegative symmetric bounded linear operator $\boldsymbol{R}$ from X^* into X is a Gaussian covariance if and only if it is nuclear.

In this paper, we first prove the following: Let X be a Banach space with an unconditional basis $\{x_k\}_{k=1}^{\infty}$. If $\boldsymbol{R} : X^* \to X$ is a nonnegative symmetric bounded linear operator such that the series $\sum_{k=1}^{\infty} <\boldsymbol{R}x_k^*, x_k^*>^{\frac{1}{2}} x_k$ converges in X, then $\boldsymbol{R}$ is nuclear.

We know from the Lemma 2.1 of [1] that the convergence of the series $\sum_{k=1}^{\infty} <\boldsymbol{R}x_k^*, x_k^*>^{\frac{1}{2}} x_k$ is necessary for $\boldsymbol{R}$ to be a covariance operator of a Gaussian measure. Therefore we have that if $\boldsymbol{R} : X^* \to X$ is a covariance operator of a Gaussian measure on a Banach space X with an unconditional basis then $\boldsymbol{R}$ is nuclear.

Second, we give an example of a nuclear operator which is not a

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Gaussian covariance in order to demonstrate that $\boldsymbol{R}: X^* \to X$ is nuclear is not sufficient for $\boldsymbol{R}$ to be a Gaussian covariance. Here again X is a Banach space with an unconditional basis.

How can we characterize the Gaussian measures on a Banach space which fails cotype and has an unconditional basis? We know that in this case the convergence of the series $\sum_{k=1}^{\infty} \langle \boldsymbol{R}x_k^*, x_k^* \rangle^{\frac{1}{2}} x_k$ is not sufficient for $\boldsymbol{R}$ to be a Gaussian covariance by finding an example of an operator $\boldsymbol{R}: X^* \to X$ such that the series $\sum_{k=1}^{\infty} \langle \boldsymbol{R}x_k^*, x_k^* \rangle^{\frac{1}{2}} x_k$ converges in X, but $\boldsymbol{R}$ is not a Gaussian covariance.

Since a nonnegative symmetric bounded linear operator $\boldsymbol{R}: X^* \to X$ is factored as $\boldsymbol{R} = A^* \circ A$, where A is a bounded linear operator from X^* into a Hilbert space H, the convergence of the series $\sum_{k=1}^{\infty} \langle \boldsymbol{R}x_k^*, x_k^* \rangle^{\frac{1}{2}} x_k$ is equivalent to the convergence of the series $\sum_{k=1}^{\infty} \|Ax_k^*\| x_k$ (cf. [8]).

We know from [6] that if a Banach space X is of cotype p for some $p < \infty$ then $(\mathbf{E}\|\sum_{k=1}^{\infty} \|Ax_k^*\| \gamma_k x_k\|^2)^{\frac{1}{2}}$ is equivalent to $(\mathbf{E}\|\sum_{k=1}^{\infty} \|Ax_k^*\| \varepsilon_k x_k\|^2)^{\frac{1}{2}}$, where $\{\gamma_k\}$ is a sequence of identically distributed independent standard Gaussian random variables and $\{\varepsilon_k\}$ is the sequence of Rademacher functions. Therefore by using the unconditionality of the basis we have $(\mathbf{E}\|\sum_{k=1}^{\infty} \|Ax_k^*\| \gamma_k x_k\|^2)^{\frac{1}{2}}$ is equivalent to $\|\sum_{k=1}^{\infty} \|Ax_k^*\| x_k\|$ when X is a Banach space which has cotype p for some $p < \infty$ and an unconditional basis $\{x_k\}$. This allows us to describe the Gaussian measures as follows: Let X be a Banach space which has cotype p and an unconditional basis $\{x_k\}$. A nonnegative symmetric bounded linear operator $\boldsymbol{R} = A^* \circ A$ from X^* into X is a covariance operator of a Gaussian measure on X if and only if $(\mathbf{E}\|\sum_{k=1}^{\infty} \|Ax_k^*\| \gamma_k x_k\|^2)^{\frac{1}{2}}$ is finite. (**)

We may conjecture that (**) is true when X has an unconditional basis even in the absence of cotype, but we don't know the answer.

Finally, we present a certain kind of operators which are γ-Radonifying.

2. Definitions

Let X denote a real Banach space and X^* denote its dual.

The canonical Gaussian cylindrical measure γ_H on a Hilbert space H

is the cylindrical measure with characteristic functional $\hat{\gamma}_H(h)=\exp\left\{-\frac{\|h\|^2}{2}\right\}$, $h\in H$.

A cylindrical measure μ on X is called a *Gaussion cylindrical measure* if there exists a Hilbert space H and a continuous linear map T from H into X such that $\mu=\gamma_H\circ T^{-1}$.

A bounded linear operator $T: H\to X$ is called *γ-Radonifying* if the Gaussian cylindrical measure $\gamma_H\circ T^{-1}$ admits extension to a tight Borel measure on the Borel field.

A bounded linear operator $T: H\to X$ is called *γ-summing* if there exists C such that for any finite subset $\{h_i\}_{i=1}^n\subset H$, $(\mathbf{E}\|\sum_{k=1}^n Th_k\gamma_k\|^2)^{\frac{1}{2}}\leq C\sup\left\{(\sum_{k=1}^n|\langle h, h_k\rangle|^2)^{\frac{1}{2}} : \|h\|=1\right\}$. The infimum of such C is denoted by $\Pi_r(T)$.

A Banach space X is of *cotype p* for some $p\geq 2$ (respectively type p for some $p\leq 2$) if there exists C such that for any finite subset $\{x_i\}_{i=1}^n\subset X$, $\left(\sum_{k=1}^n\|x_k\|^p\right)^{\frac{1}{p}}\leq C\left(\mathbf{E}\|\sum_{k=1}^n x_k\varepsilon_k\|^p\right)^{\frac{1}{p}}$, (respectively $\geq$).

Let X and Y be Banach spaces. For $0<p<\infty$, a bounded linear operator $T: X\to Y$ is called *absolutely p-summing* if there exists C such that for any finite subset $\{x_i\}_{i=1}^n\subset X$, $\left(\sum_{k=1}^n\|Tx_k\|^p\right)^{\frac{1}{p}}\leq C\sup\left\{\left(\sum_{k=1}^n|\langle x^*, x_k\rangle|^p\right)^{\frac{1}{p}} : \|x^*\|\leq 1\right\}$. The infimum of such C is the absolutely p-summing norm of T and denoted by $\Pi_p(T)$.

For $1\leq p<\infty$, a bounded linear operator $T: X\to Y$ is called p-integral if there is a probability measure μ and two operators $\boldsymbol{R}: X\to L_\infty(\mu)$, $S: L_p(\mu)\to Y^{**}$ such that $JT=S\beta\boldsymbol{R}$, where $\beta: L_\infty(\mu)\to L_p(\mu)$ is inclusion and $J: Y\to Y^{**}$ is the canonical embedding of a space Y into its bidual. The p-integral norm of T is defined by $i_p(T)=\inf\{\|\boldsymbol{R}\|\|S\|\}$, where the infimum is taken over all such factorizations of JT.

A bounded linear operator $T: X\to Y$ is called nuclear if there exist sequences $\{x_n^*\}$ in X^* and $\{y_n\}$ in Y such that $\sum_{n=1}^\infty\|x_n^*\|\|y_n\|<\infty$ and $T(x)=\sum_{n=1}^\infty x_n^*(x)y_n$ for all $x\in X$. The nuclear norm of T is defined by $\nu(T)=\inf\left\{\sum_{n=1}^\infty\|x_n^*\|\|y_n\|\right\}$, where the infimum is taken over all sequences $\{x_n^*\}$ and $\{y_n\}$ such that $x_n^*\in X^*$, $y_n\in Y$ and $T(x)=\sum_{n=1}^\infty x_n^*(x)y_n$ for all $x \in X$.

3. Results and examples

The proof of our theorem below is based on the following two lemmas.

LEMMA 1. *Let X be a Banach space with an unconditional basis $\{x_k\}_{k=1}^{\infty}$. If $\boldsymbol{R} : X^* \to X$ is a nonnegative symmetric bounded linear operator such that the series $\sum_{k=1}^{\infty} <\boldsymbol{R}x_k^*, x_k^*>^{\frac{1}{2}} x_k$ converges in X and $A : X^* \to H$ is the operator satisfying $R = A^*A$ then there exist bounded operators $A_1 : X^* \to l_1$ and $A_2 : l_1 \to H$ such that $A = A_2A_1$.*

Proof. Since $\{x_k\}$ is an unconditional basis, for each $x^* \in X^*$, the series $x^*(\sum \|Ax_k^*\| x_k) = \sum \|Ax_k^*\| x^*(x_k)$ converges unconditionally. Then $\sum \|Ax_k^*\| x^*(x_k)$ also converges absolutely and so $\{\|Ax_k^*\| x^*(x_k)\}_{k=1}^{\infty}$ belongs to l_1. Hence we can take a subsequence $\{n_p\}_{p=1}^{\infty}$ and set $\lambda_i = p$ for $n_{p-1} < i \leq n_p$, $p \geq 1$ so that $\{\lambda_k x^*(x_k) \|Ax_k^*\|\}_{k=1}^{\infty}$ also belongs to l_1.

In fact, since $A^* : H \to X$, A is also continuous from $(X^*, \sigma(X^*, X))$ into $(H, \sigma(H, H))$. So $Ax^* = \sum x^*(x_k) Ax_k^*$, where the series converges in weak topology on H unconditionally.

Now define an operator $A_1 : X^* \to l_1$ by $A_1 x^* = \{\lambda_i x^*(x_i) \|Ax_i^*\|\}_{i=1}^{\infty}$, where $\lambda_i = p$ for $n_{p-1} < i \leq n_p$, $p \geq 1$ and an operator $A_2 : l_1 \to H$ by $A_2(\{\alpha_n\}) = \sum (\lambda_n)^{-1} \alpha_n h_n$, where $h_n = (Ax_n^*) \|Ax_n^*\|^{-1}$ if $Ax_n^* \neq 0$ and 0 otherwise. It follows from the construction of A_1 and A_2 that $A = A_2A_1$. It is obvious that $\|A_2\| \leq 1$ and so A_2 is a bounded operator. Let $T : C_0 \to X$ be the operator defined by $T(\{\beta_i\}) = \sum \lambda_i \beta_i \|Ax_i^*\| x_i$, where $\lambda_i = p$ for $n_{p-1} < i \leq n_p$, $p \geq 1$. Then $\|T\| \leq \| \sum \lambda_i \|Ax_i^*\| x_i \| < \infty$ and $T^* = A_1$. Hence A_1 is a bounded operator. This completes the proof of Lemma 1.

REMARKS.

1. It is easy to see that $A_2^*(H) \subseteq C_0$ by computing A_2^*.
2. Since $A_2^{**} = A_2$, $A_2^*A_2 : l_1 \to C_0$ is a nonnegative symmetric bounded linear operator.
3. We have a factorization $\boldsymbol{R} : X^* \xrightarrow{A_1} l_1 \xrightarrow{A_2} H \xrightarrow{A_2^*} C_0 \xrightarrow{A_1^*} X$.
4. Let $\{e_k\}$ be the unit vector basis of C_0 and let $\{e_k^*\}$ be the sequence of biorthogonal functionals associated with the basis $\{e_k\}$. The series $\sum <A_2^*A_2 e_k^*, e_k^*>^{\frac{1}{2}} e_k$ converges in C_0.

Now we state the known results which are needed in proving Lemma 2.

1. Let X and Y be Banach spaces and X or Y be finite dimensional. For every bounded linear operator T from X into Y, we have $\nu(T)=i_1(T)$. (cf. [7]).
2. Let X be a Banach space. A bounded linear operator T from X into C_0 is absolutely summing if and only if it is integral, and $\Pi_1(T)=i_1(T)$. (cf. [7]).
3. (Grothendieck, Lindenstrauss, Pelczynski [5]). Every bounded linear operator T from l_1 into l_2 is absolutely summing and $\Pi_1(T)\leq K_G\|T\|$, where K_G is Grothendieck constant.

LEMMA 2. *Let $S: l_1 \rightarrow C_0$ be a nonnegative symmetric bounded linear operator such that the series $\sum_{k=1}^{\infty} <Se_k^*, e_k^*>^{\frac{1}{2}} e_k$ converges in C_0. Then S is nuclear.*

Proof. Since S is a nonnegative symmetric bounded linear operator from $C_0^*=l_1$ into C_0, there exists a factorization $S=TT^*$ such that $S: l_1 \xrightarrow{T^*} l_2 \xrightarrow{T} C_0$. Therefore the convergence of $\sum <Se_k^*, e_k^*>^{\frac{1}{2}} e_k$ is equivalent to that of $\sum \|T^*e_k^*\| e_k$ in C_0. Let $\|T^*e_k^*\|=\delta_k$. Then δ_k converges to 0. Choose a subsequence $\{n_k\}_{k=1}^{\infty}$ so that $\max_{j>n_k} \delta_j < 2^{-k}$, $k\geq 1$.

Define projections $P_l: l_1 \rightarrow l_1^{n_{l+1}-n_l}$, $P_l: l_2 \rightarrow l_2^{n_{l+1}-n_l}$ by $P_l(e_j)=0$ if $j\leq n_l$ or $j>n_{l+1}$ and $P_l(e_j)=e_j$ if $n_l<j\leq n_{l+1}$. We set $n_0=0$. It is easy to see that $T^*=\sum_{l=0}^{\infty} P_l T^* P_l$. Let $S_l=P_l T^* P_l$. Then for every $x\in l_1$, $\|S_l(x)\| \leq \| \sum_{j=n_{l+1}}^{n_{l+1}} x(e_j) T^*(e_j)\| \leq 2^{-l}\|x\|$. Hence $\|S_l\|\leq 2^{-l}$ for $l\geq 1$. Note that $\|S_0\|\leq \|T^*\|$ and $\|S_l^*\|\leq \|T\|$ for $l\geq 0$. Since $S_l^* S_k: l_1 \rightarrow C_0$ is a finite rank operator, it follows from the above results 1, 2 and 3 that $\nu(S)=\nu\Big(\Big(\sum_{l=0}^{\infty} S_l^*\Big)\Big(\sum_{k=0}^{\infty} S_k\Big)\Big) \leq \sum_{l,k=0}^{\infty} \|S_l^*\| K_G \|S_k\| \leq K_G \|T\| \Big(\sum_{k=1}^{\infty} 2^{-k} + \|T^*\|\Big) < \infty$, where K_G is Grothendieck constant. This proves that S is nuclear.

THEOREM. *Let X be a Banach space with an unconditional basis $\{x_k\}_{k=1}^{\infty}$. If $\boldsymbol{R}: X^* \rightarrow X$ is a nonnegative bounded linear operator such that the series $\sum_{k=1}^{\infty} <\boldsymbol{R}x_k^*, x_k^*>^{\frac{1}{2}} x_k$ converges in X then $\boldsymbol{R}$ is nuclear.*

Proof. By Remark 3, $\boldsymbol{R}$ can be factored as $\boldsymbol{R}: X^* \xrightarrow{A_1} l_1 \xrightarrow{A_2} H \xrightarrow{A_2^*} C_0 \xrightarrow{A_1^*}$

X. From Remark 1, 2 and 4, we see that an operator $A_2^*A_2$ satisfies conditions of Lemma 2. Thus $A_2^*A_2$ is nuclear and so $\boldsymbol{R}$ is nuclear.

Corollary. *If* $\boldsymbol{R}: X^* \to X$ *is a covariance operator of a Gaussian measure on a Banach space* X *with an unconditional basis* $\{x_k\}$ *then* $\boldsymbol{R}$ *is nuclear.*

Proof. It follows from the Lemma 2.1 of [1] that if $\boldsymbol{R}$ is a covariance operator of a Gaussian measure on X then the series $\sum \langle \boldsymbol{R}x_k^*, x_k^* \rangle^{\frac{1}{2}} x_k$ converges in X. Hence by the theorem, $\boldsymbol{R}$ is nuclear.

Combining our theorem and W. Linde's and A. Pietsch's result [4], which states that a bounded linear diagonal operator $S: l_2 \to l_\infty$ of the form $S(e_n) = \delta_n e_n$ where $\delta_1 \geq \delta_2 \geq \cdots \geq 0$, is γ–summing if and only if $\sup\limits_n \delta_n \sqrt{\log(n+1)} < \infty$ $(*)$, we find an example of a nuclear operator which is not a Gaussian covariance.

Example. Let T be a bounded linear operator from l_1 into C_0 defined by $T(e_i) = \delta_i e_i$, where $\delta_1 \geq \delta_2 \geq \cdots \geq 0$. If the sequence $\{\sqrt{\delta_k}\}$ converges to 0, but the sequence $\{\sqrt{\delta_k}\sqrt{\log(k+1)}\}$ diverges then T is nuclear, but not a Gaussian covariance.

Proof. Note that $\{e_k\}$ is an unconditional basis of C_0 and T is a nonnegative symmetric bounded linear operator with $\|\sum \langle Te_k, e_k \rangle^{\frac{1}{2}} e_k\| = \sqrt{\delta_1} < \infty$. Hence by the theorem T is nuclear. Now T is factored as $T = A^*A$, where A is a bounded linear operator from l_2 into C_0 defined by $Ae_i = \sqrt{\delta_i} e_i$. By $(*)$, the condition $\lim\limits_{k\to\infty} \sqrt{\delta_k}\sqrt{\log(k+1)} = \infty$ implies that A is not γ–summing. Therefore A is not γ–Radonifying, that is, T is not a Gaussian covariance.

This example shows that the fact that a nonnegative symmetric bounded linear operator $\boldsymbol{R}$ from X^* into X is nuclear is not sufficient for $\boldsymbol{R}$ to be a Gaussian covariance when X is a Banach space with an unconditional basis.

Is it true that if $\boldsymbol{R}: X^* \to X$ is a nonnegative symmetric bounded linear operator such that the series $\sum \langle \boldsymbol{R}x_k^*, x_k^* \rangle^{\frac{1}{2}} x_k$, where $\{x_k\}$ is an unconditional basis of X, converges in X then $\boldsymbol{R}$ is a Gaussian covariance? Again using $(*)$, we give a counterexample. The Banach space $\left(\sum_{n=1}^{\infty} \oplus l_{\infty}^{n}\right)_2$ is defined as follows: $\left(\sum_{n=1}^{\infty} \oplus l_{\infty}^{n}\right)_2 = \{\{x_n\} : x_n \in l_{\infty}^{n}$ for each n, $\|\{x_n\}\| = \left(\sum_{n=1}^{\infty} \|x_n\|^2\right)^{\frac{1}{2}} < \infty\}$. Take X as the Banach space $\left(\sum_{n=1}^{\infty} \oplus l_{\infty}^{n}\right)_2$. Then $X^* = \left(\sum_{n=1}^{\infty} \oplus l_1^{n}\right)_2$. Let $\{e_{n,m}\}_{m=1,n=1}^{n\ \ \infty}$ be the natural basis of X and let $\{e_{n,m}^*\}_{m=1,n=1}^{n\ \ \infty}$ be the sequence of biorthogonal functionals associated with the basis $\{e_{n,m}\}$. Let $\{\sigma_k\}$ be a sequence of real numbers such that $\sigma_1 \geq \sigma_2 \geq \cdots \geq 0$, $\lim_{k\to\infty} \sigma_k = 0$ and $\lim_{k\to\infty} \sigma_k \log(k+1) = \infty$. Now define a bounded linear operator $R_{j,m}$ from l_1^m into l_{∞}^m by $R_{j,m}(e_{m,n}^*) = \sigma_{j+n-1} e_{m,n}$, $n = 1, 2, \cdots m$. Since $\lim_{n\to\infty} \sigma_{j+n-1} \log(n+1) = \infty$, by (*) $\sqrt{R_{j,m}}$ is not γ-summing. Therefore, for each j, there exists m such that $\Pi_r(\sqrt{R_{j,m}}) > 2^j$.

Let P_n be a projection from $(\sum \oplus l_1^n)_2$ onto l_1^n defined by $P_n(\{x_m\}) = x_n$ and J_n be a injection from l_{∞}^n into $(\sum \oplus l_{\infty}^n)_2$ defined by $J_n(x_n) = \{x_m\}$, where $x_m = 0$ if $m \neq n$. Denote O_n by a zero map from l_1^n into l_{∞}^n. Choose a subsequence of σ_j's, say σ_{l_k} such that $\sum_{k=1}^{\infty} \sigma_{l_k} < \infty$. Define a nonnegative symmetric bounded linear operator S from X^* into X by $S = \sum_{k=1}^{\infty} J_{m_{l_k}} R_{l_k, m_{l_k}} + \sum_{k \notin (m_{l_1}, m_{l_2}, \cdots)} J_k O_k P_k$. It is easy to see that $\|\sum_{n=1}^{\infty} \sum_{m=1}^{n} <Se_{n,m}^*, e_{n,m}^*>^{\frac{1}{2}} e_{n,m}\|^2 = \sum_{k=1}^{\infty} \sigma_{l_k} < \infty$. Hence the series $\sum_{n,m} <Se_{n,m}^*, e_{n,m}^*>^{\frac{1}{2}} e_{n,m}$ is convergent in X. S is factored as $S = AA^*$, where A is a bounded linear operator from H into X. Since $\Pi_r(A) \geq \Pi_r(\sqrt{R_{l_k, m_{l_k}}}) < 2^{l_k}$ for all k, A is not γ-summing. Hence A is not γ-Radonifying, that is, S is not a Gaussian covariance.

In fact, by using this example we have the following proposition which shows that the convergence of the series $\sum_{k=1}^{\infty} <\boldsymbol{R}x_k^*, x_k^*>^{\frac{1}{2}} x_k$ is not sufficient for $\boldsymbol{R}$ to be a Gaussian covariance when a Banach space fails cotype and has an unconditional basis $\{x_k\}$.

Proposition. *Let X be a Banach space which fails cotype and has an unconditional basis $\{x_k\}_{k=1}^{\infty}$. There is a nonnegative symmetric bounded linear operator S from X^* into X such that the series $\sum_{k=1}^{\infty} <Sx_k^*, x_k^*>^{\frac{1}{2}} x_k$ is convergent in X, but S is not a Gaussian covariance.*

Proof. Let $\{\sigma_k\}_{k=1}^{\infty}$ be a sequence of real numbers such that $\sigma_1 \geq \sigma_2$

$\geq \cdots \geq 0$, $\lim_{k\to\infty} \sigma_k = 0$ and $\lim_{k\to\infty} \sigma_k \log(k+1) = \infty$. Choose a subsequence of σ_j's, say σ_{q_k}, such that $\sum_{k=1}^{\infty} \sqrt{\sigma_{q_{k-1}+1}} < \infty$. Since Banach space X fails cotype and has an unconditional basis $\{x_k\}$, using Maurey and Pisier's theorem [6], for all n, we can find n vectors $\{y_i\}_{i=1}^{n}$ in X such that $y_i = \sum_{j=p_{i-1}+1}^{p_i} \eta_j x_j$, $\eta_j \geq 0$ for all j, $\|y_i\| = 1$ for $i = 1, 2, \cdots n$ and $\max_{1\leq i\leq n} |C_i| \leq \|\sum_{i=1}^{n} C_i y_i\| \leq (1+\varepsilon) \max_{1\leq i\leq n} |C_i|$ for all $(C_1, C_2, \cdots C_n) \in \boldsymbol{R}^n$ (**).

In particular, we find q_1 vectors $\{y_i\}_{i=1}^{q_1}$ in X satisfying (**). Let $Y_1 = [y_l : l \leq l \leq q_1]$. Note that if a Banach space X fails cotype and has an unconditional basis $\{x_k\}$ then for any N, $[x_i]_{i>N}$ also fails cotype. Hence we can find $q_2 - q_1$ vectors $\{y_i\}_{i=q_1+1}^{q_2}$ in $\overline{X - Y_1}$ satisfying(**). In this way, we find $y_i = \sum_{j=p_{i-1}+1}^{p_i} \eta_j x_j$, $\eta_j \geq 0$ for all j and $y_i^* = \sum_{j=p_{i-1}+1}^{p_i} \theta_j x_j^*$ with $\|y_i\| = \|y_i^*\| = 1$, $\langle y_j^*, y_i \rangle = \delta_{ij}$ such that $\max_{q_{k-1}+1\leq i\leq q_k} |C_i| \leq \|\sum_{i=q_{k-1}+1}^{q_k} C_i y_i\| \leq (1+\varepsilon) \max_{q_{k-1}+1\leq i\leq q_k} |C_i|$ for all $(C_{q_{k-1}+1}, \cdots, C_{q_k}) \in \boldsymbol{R}^{q_k - q_{k-1}}$.

Now define a no negative symmetric operator S from X^* into X by $S(x^*) = \sum_{l=1}^{\infty} \sigma_l < x^*, y_l > y_l$ for all $x^* \in X^*$. Let $Y_j = [y_l : q_{j-1} < l \leq q_j]$ and $Y_j^* = [y_l^* : q_{j-1} < l \leq q_j]$. Here we set $q_0 = 0$. $S|_{Y_j}^*$ is an operator from Y_j^* into Y_j and $S|_{Y_j}^*(y_l^*) = \sigma_l y_l$ for $q_{j-1} < l \leq q_j$, $\|S\| \leq \sum_{j=1}^{\infty} \|S|_{Y_j}^*\| \leq (1+\varepsilon)\left(\sum_{j=1}^{\infty} \sqrt{\sigma_{q_{j-1}+1}}\right)^2 < \infty$. Hence S is a bounded operator. Define a diagonal operator $\tilde{S}_j$ from $l_1^{q_j - q_{j-1}}$ into $l_\infty^{q_j - q_{j-1}}$ by $\tilde{S}_j(e_l^*) = \sigma_l e_l$, $q_{j-1} < l \leq q_j$. Since $\lim_{l\to\infty} \sqrt{\sigma_l}\sqrt{\log(l+1)} = \infty$, $\Pi_r(\sqrt{\tilde{S}_j}) \to \infty$ as $j \to \infty$. But Y_j^* is isomorphic to $l_1^{q_j - q_{j-1}}$ and Y_j is isomorphic to $l_\infty^{q_j - q_{j-1}}$. So $\Pi_r(\sqrt{S|_{Y_j}^*}) \to \infty$, $j \to \infty$. Because $\Pi_r(\sqrt{S}) \geq \Pi_r(\sqrt{S}|_{Y_j}^*)$ for all j, $\sqrt{S}$ is not γ-summing. Therefore S is not a Gaussian covariance.

If $p_{j-1} < l \leq p_j$ then $< Sx_l^*, x_l^* >^{\frac{1}{2}} = \sqrt{\sigma_j} \eta_l$. Then

$$\|\sum_{l=1}^{\infty} < Sx_l^*, x_l^* >^{\frac{1}{2}} x_l\| = \|\sum_{j=1}^{\infty} (\sqrt{\sigma_j} y_j)\| \leq \sum_{k=1}^{\infty} \|\sum_{q_{k-1}+1}^{q_k} \sqrt{\sigma_j} y_j\|$$
$$\leq (1+\varepsilon) \sum_{k=1}^{\infty} \sqrt{\sigma_{q_{k-1}+1}} < \infty.$$

Hence the series $\sum < Sx_l^*, x_l^* >^{\frac{1}{2}} x_l$ is convergent in X.

In the rest of this paper, we present a certain kind of operators which are γ-Radonifying. We need lemma to find such operators.

LEMMA. *Let Δ be a bounded linear operator from C_0 into C_0 defined by $\Delta(e_i)=\delta_i e_i$, where $\delta_1 \geq \delta_2 \geq \cdots \geq 0$.*
(a) If $\left(\mathbf{E}\|\sum_{i=1}^{\infty} \delta_i e_i \gamma_i\|^2\right)^{\frac{1}{2}}$ is finite then Δ is γ-summing.
(b) If there exists a constant $M>0$ such that $\delta_i\sqrt{\log(i+1)} \leq M$ for all i then Δ is γ-summing.

Proof. (a) By definition, Δ is γ-summing if and only if for every bounded linear operator A from l_2 into C_0, $(\mathbf{E}\|\sum(\Delta A)(h_i)\gamma_i\|^2)^{\frac{1}{2}}$ is finite, where $\{h_i\}$ is an orthonormal system in l_2. Let A be a bounded linear operator from l_2 into C_0 defined by $Ah_i=\sum_{k=1}^{\infty}\eta_{ik}e_k$, $i=1, 2, \cdots$. Without loss of generality, we can assume that $\|A\|=1$. Since $\|A\|=\sup_k\left(\sum_{i=1}^{\infty}|\eta_{ik}|^2\right)^{\frac{1}{2}}$, we have $\sum_{i=1}^{\infty}\eta_{ik}^2 \leq 1$ for each k. By definition, $\sum(\Delta A)(h_i)\gamma_i=\sum\delta_k\Gamma_k e_k$, where $\Gamma_k=\sum_{i=1}^{\infty}\eta_{ik}\gamma_i$. Each Γ_k is a scalar Gaussian random variable with mean zero and variance $\sum_{i=1}^{\infty}\eta_{ik}^2 \leq 1$. Next, $\|\sum(\Delta A)(h_i)\gamma_i\|=\max_{1\leq k\leq\infty}|\delta_k\Gamma_k|$ and $\|\sum\delta_i\left(\sum_k\eta_{ki}^2\right)^{\frac{1}{2}}\gamma_i e_i\|=\max_{1\leq i\leq\infty}|\delta_i\left(\sum_k\eta_{ki}^2\right)^{\frac{1}{2}}\gamma_i|$. Let $X_k=\delta_k(\sum_i\eta_{ik}^2)^{\frac{1}{2}}\gamma_k$ and $Y_k=\delta_k\Gamma_k$. Then, for each k, X_k and Y_k are scalar Gaussian random variables with mean zero and variance $\delta_k^2\sum_i\eta_{ik}^2$. The X_k's are independent, but Y_k's are not independent. Hence $0=\mathbf{E}X_iX_j \leq \mathbf{E}Y_iY_j$ if $i\neq j$. Thus Slepian's lemma [3] gives us that $P[\max_k Y_k>t] \leq P[\max_k X_k>t]$. But, for each k, X_k and Y_k are symmetric and so $\mathbf{E}(-X_i)(-X_j) \leq \mathbf{E}(-Y_i)(-Y_j)$. Again, it follows rom Slepian's lemma that $P[\max_k(-Y_k)>t] \leq P[\max_k(-X_k)>t]$. Therefore $P[\max_k|Y_k|>t] \leq P[\max_k|X_k|>t]$. Hence we have $\mathbf{E}[(\max_k|Y_k|)^2] \leq \mathbf{E}[(\max_k|X_k|)^2]$. Therefore,

$$\left(\mathbf{E}\|\sum(\Delta A)(h_i)\gamma_i\|^2\right)^{\frac{1}{2}} \leq \left(\mathbf{E}\|\sum\delta_i\gamma_i e_i\|^2\right)^{\frac{1}{2}}<\infty.$$

Hence Δ is γ-summing.

(b) It follows from lemma 6 of [4] that the boundedness of $\delta_i\sqrt{\log(i+1)}$, for all i, forces $\left(\mathbf{E}\|\sum\delta_i\gamma_i e_i\|^2\right)^{\frac{1}{2}}<\infty$. Hence, by (a), Δ is γ-summing.

PROPOSITION. *Let X be a Banach space which has an unconditional basis $\{x_k\}$ and contains no subspace isomorphic to C_0. Let Δ be a bounded* *linear operator from C_0 into X defined by $\Delta(e_i)=\delta_i x_i$, where $\delta_1 \geq \delta_2 \geq \cdots \geq 0$. If $\tilde{\Delta}(e_i)=\delta_i\sqrt{\log(i+1)}\,x_i$ defines a bounded linear operator from C_0 into X then for every bounded linear operator A from l_2 into C_0, the bounded linear operator $\Delta \circ A$ from l_2 into X is γ-Radonifying.*

Proof. Let $\wedge$ be a bounded linear operator from C_0 into C_0 defined by $\wedge(e_i)=(\sqrt{\log(i+1)})^{-1}e_i$ so that Δ factors as $\Delta=\tilde{\Delta}\wedge$. By the lemma (b), $\wedge$ is γ-summing and so Δ is also γ-summing. Since X contains no subspace isomorphic to C_0, by theorem 1 of [2], we have that Δ is γ-Radonifying.

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Dongguk University
Seoul 100-715, Korea