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THE EXTENSION OF SOLUTIONS FOR THE CAUCHY PROBLEM IN THE COMPLEX DOMAIN

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Introduction

In [4], J. Leray introduced the notion of partial hyperbolicity to characterize the operators for which the non-characteristic Cauchy problem is solvable in the Gevrey class for any data which are holomorphic in a part of variables $x''=(x_2, \dots, x_l)$ in the initial hyperplane $x_1=0$. A linear partial differential operator is called partially hyperbolic modulo the linear subvarieties $S: x''=$constant if the equation $P_m(x, \zeta_1, \xi')=0$ for ζ_1 has only real roots when ξ' is real and $\xi''=0$, where P_m is the principal symbol of P.

Limiting to the case of operators with constant coefficients, A. Kaneko proposed a new sharper condition when S is a hyperplane [3].

In this paper, we generalize this condition to the case of general linear subvariety S and show that it is sufficient for the solvability of Cauchy problem for the hyperfunction Cauchy data which contains variables parallel to S as holomorphic parameters.

Let $P(D)$ be an m-th order linear partial diffierential operator in $\boldsymbol{R}^n$ with constant coefficients, and let $P_m(D)$ be its principal part. Assume that $x_1=0$ is non-characteristic with respect to P. We use the following notation for the separation of the independent variables; $x=(x_1, x')=(x_1, x'', x''')$ with $x''=(x_2, \dots, x_l)$ and similar notation for the complexification $z=x+\sqrt{-1}y$ for the dual variables $\zeta=\xi+\sqrt{-1}\eta$.

We let Γ be a convex open cone in $\boldsymbol{R}^{n-l}$ and Δ be a convex open cone such that $\Delta\subset\subset\Gamma$, i.e., $\bar{\Delta}$ is a compact subset of Γ.

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LEMMA. *Consider the holomorphic Cauchy problem*

$$(2.1)\qquad \begin{cases} P(D)F(z)=0 \\ \left(\dfrac{\partial}{\partial z_1}\right)^j F(z)\,|_{z_1=0}=F_j(z'), \quad j=0,\dots,m-1 \end{cases}$$

The holomorphic data $F_j(z')$ are given on a domain of the form $\{z'\in C^{n-1}|\,|x'|<A,\ y'''\in\Gamma,\ |y'|<B\}$. Also, for some constant $b, c>0$, we have

$$(2.2)\qquad -Im\,\zeta_1\le b|Im\,\zeta'''|+c|\zeta''|$$

if $Re\,\zeta'''\in\Delta^\circ$.

Then the solution can be continued onto the domain

$$W=\{z\in C^n|\,0\le x_1<\delta,\ |x|<A',\ \lambda|y_1|<dis(y'',\partial\Gamma),\ |y'|<B'\}$$

where A', B', λ and δ are suitable positive constant.

Proof. First note that by the Cauchy-Kowalevsky theorem, the solution exists on a domain

$$\widetilde{W}=\{z\in C^n|\,|z_1|<k\,\mathrm{dis}(y''',\partial\Gamma),\ \ |x'|<A/2,\ |y''|<B/2,\ y'''\in\Gamma,\ |y'''|<B/2\}$$

where k is a positive constant. Starting from this open set, we may use the method of Bony-Schapira [1]. Choose

$$z_0=(t+\sqrt{-1}s,0,\dots,0,\sqrt{-1}y_0)$$

where $t>0$, $y_0\in\Gamma$, $|y_0|<\varepsilon$. If every real characteristic hyperplane passing through this point intersects $\widetilde{W}$, then the solution $F(z)$ can be continued up to the interior of $\mathrm{ch}[\{z_0\cup\widetilde{W}\}]$.

A characteristic hyperplane passing through z_0 is expressed by the following equation

$$(2.3)\qquad -\mathrm{Re}\langle z-z_0,\sqrt{-1}\zeta\rangle=x\cdot\eta+y\cdot\xi-t\eta_1-s\xi_1-y_0\cdot\xi'''=0$$

where $\zeta=\xi+\sqrt{-1}\eta$ satisfies $P_m(\zeta)=0$. The fact that $P_m(\zeta)=0$ and the non-characteristic assumption imply that there exists $M>0$ such that

$$|\zeta_1|\le M|\zeta'|.$$

We consider the following two case.

(1) Case $|\xi'|\le|\eta'|$.

The point

$$x_1=0,\ \ x'=\frac{t\eta_1+s\xi_1}{|\eta'|^2}\eta',\ \ y_1=y''=0,\ \ y'''=y_0$$

satisfies (2.3). Since we have

$$|\xi_1|\le|\zeta_1|\le M|\zeta'|\le\sqrt{2}M|\eta'|$$

and similarly, $|\eta_1|\le\sqrt{2}M|\eta'|$, this point is contained in $\widetilde{W}$ provided that

$$|x'| \leq \frac{t|\eta_1| + |s||\xi_1|}{|\eta'|} \leq \sqrt{2}M(t+|s|) < A/2$$

and $\varepsilon < B/2$.

(2) Case $|\xi'| \geq |\eta'|$.

First suppose that $|\xi'''| \leq |\xi''|$.

The point

$$x=0, \quad y_1=0, \quad y''=\frac{t\eta_1+s\xi_1}{|\xi''|^2}\xi'', \; y'''=y_0$$

satisfies (2. 3). Since we have

$$|\xi_1| \leq M|\zeta'| \leq \sqrt{2}M|\xi'| \leq 2M|\xi''|$$

and $|\eta_1| \leq 2M|\xi''|$, this point is contained in $\widetilde{W}$ provided that

$$|y''| \leq \frac{t|\eta_1| + |s||\xi_1|}{|\xi''|} \leq 2M(t+|s|) < B/2$$

and $\varepsilon < B/2$.

Next consider $\zeta \in C^n$ such that $|\xi'''| \geq |\xi''|$ and Re $\zeta''' \notin \Delta^\circ$. Then there exists $\gamma \in \Gamma$ with $|\gamma|=1$ such that

$$\xi''' \cdot \gamma < -C|\xi'''|$$

where C is a constant (independent of ξ'''). The point

$$x=0, \quad y_1=y''=0, \quad y'''=y_0+\frac{t\eta_1+s\xi_1}{\xi''' \cdot \gamma}\gamma$$

satisfies (2. 3). Without loss of generality, we can assume that $t\eta_1 + s\xi_1 \leq 0$. (If this is not the case, then we can replace ζ by $-\zeta$.)

Since we have

$$|\xi_1| \leq M|\zeta'| \leq 2M|\xi'''|$$

and, similarly, $|\eta_1| \leq 2M|\xi'''|$, this point is contained in $\widetilde{W}$ provided that

$$|y'''| \leq |y_0| + \frac{t|\eta_1| + |s||\xi_1|}{|\xi''' \cdot \gamma|} \leq \varepsilon + \frac{2M(t+|s|)}{C} < B/2.$$

Finally, consider $\zeta \in C^n$ such that $|\xi'''| \geq |\xi''|$ and Re $\zeta''' \in \Delta^\circ$. If $\eta_1 > 0$, then we have

$$s\xi_1 \leq t\eta_1 + s\xi_1 \leq 0.$$

Therefore, the point

$$x=0, \quad y_1=\frac{t\eta_1+s\xi_1}{\xi_1}, \quad y''=0, \quad y'''=y_0$$

satisfies (2. 3). This point is contained in $\widetilde{W}$ provided that

$$|z_1| \leq |s| < k \operatorname{dis}(y_0, \partial\Gamma)$$

and $\varepsilon<B/2$.

If $\eta_1\leq 0$, by hypothesis, we have a decomposition of form

$$\eta_1=\alpha+\beta+\gamma$$

where $|\alpha|\leq b|\eta'''|$, $|\beta|\leq c|\xi''|$, $|\gamma|\leq c|\eta''|$.

The point

$$x_1=0,\quad x''=\frac{t\gamma}{|\eta''|^2}\eta'',\quad x'''=\frac{t\alpha}{|\eta'''|^2}\eta''',$$

$$y_1=s,\quad y''=\frac{t\beta}{|\xi''|^2}\xi'',\quad y'''=y_0$$

satisfies (2. 3). If $|x'|\leq t(b^2+c^2)^{\frac{1}{2}}<A/2$, $|y''|\leq tC<B/2$ and $|z_1|=|s|\leq k\ \mathrm{dis}(y_0,\ \partial\Gamma)$, $\varepsilon<B/4$, then this point is contained in $\widetilde{W}$.

Now by (1) and (2), if we choose $K, s, \varepsilon>0$ such that $t+|s|<K$, $|s|\leq k\ \mathrm{dis}(y_0, \partial\Gamma)$, $\varepsilon<B/4$ where

$$K=\min\left\{\frac{B}{4M},\ \frac{BC}{8M},\ \frac{A}{2\sqrt{2}M},\ \frac{A}{2(b^2+c^2)^{\frac{1}{2}}},\ \frac{B}{2C}\right\},$$

then the solution $F(z)$ can be continued up to $\mathrm{ch}[\{z_0\cup\widetilde{W}\}]$ for every $t>0$, $\varepsilon>0$. When we let them vary under these conditions and make the unions of these convex domains, we clearly obtain a domain of the form W.

THEOREM. *Assume that for some constant $b, c>0$.*

$$-Im\,\zeta_1\leq b|Im\,\zeta'''|+c|\zeta''|,$$

if $Pm(\zeta)=0$ and $Re\,\zeta'''\in\Delta^\circ$. Assume that the hyperfunction data $u_j(x')$, $j=0, \dots, m-1$, can be expressed as the boundary values of functions $F_j(z')$ holomorphic in $\{\boldsymbol{R}^{n-1}\times i(\boldsymbol{R}^{l-1}\times\Gamma)\}\cap\{|z'|<\delta\}$. Then on a neighborhood of the origin we can solve the following boundary value problem

$$\begin{cases} P(D)u=0 \\ \left(\dfrac{\partial}{\partial x_1}\right)^j u|_{x_1\to+0}=u_j(x'),\quad j=0, \dots, m-1. \end{cases}$$

Proof. With the initial data $F_j(z')$, we are going to solve the holomorphic Cauchy problem (2. 1). Put $A=B=\delta/\sqrt{2}$. Then by Lemma, the holomorphic solution $F(z)$ can be continued to the domain

$$W=\{z\in C^n\,|\,0\leq x_1<A',\ |x'|\leq A',\ \lambda|y_1|<\mathrm{dis}(y''', \partial\Gamma),\ |y''|<B',\ |y'''|<B'\}$$

which is a wedge with its edge tangent to the real axis. Thus $F(z)$ continued there defines a hyperfunction solution $u(x)$ of $P(D)u=0$ on

$x_1>0$ locally on a neighborhood of the origin. Moreover, by [3, Lemma 2.6], the boundary values of u agree with the given data u_j. Therefore the proof is complete.

Similarly, for the boundary value problem to $x_1<0$, we can prove the sufficient condition

$$Im\,\zeta_1\le b|Im\,\zeta'''|+c|\zeta''| \text{ if } P_m(\zeta)=0,\ \mathrm{Re}\,\zeta'''\in\Delta^\circ$$

for some constant $b, c>0$.

Corollary 1. *Assume that for some constant $b, c>0$,*

$$|Im\,\zeta_1|\le b|Im\,\zeta'''|+c|\zeta''|$$

if $Pm(\zeta)=0$ and $Re\,\zeta'''\in\Delta^\circ$. Assume that the hyperfunction data $u_j(x')$ can be expressed as the boundary values of functions $F_j(z')$ holomorphic in $\{\boldsymbol{R}^{n-1}\times i(\boldsymbol{R}^{l-1}\times\Gamma)\}\cup\{|z'|<\delta\}$. Then the Cauchy problem

$$\begin{cases} P(D)u=0 \\ \left(\dfrac{\partial}{\partial x_1}\right)^j u|_{x_1=0}=u_j(x'),\quad j=0,\dots,m-1 \end{cases}$$

admits a hyperfunction solution which contains the same holomorphic parameters.

Corollary 2. *Let $\Delta\subset S^{n-l}$ be an open subset. Assume that the data $u_j(x')$ contain x'' as holomorphic parameters and satisfy*

$$S.S.\,u_j\subset\subset\boldsymbol{R}^{n-1}\times\Delta dx'''.$$

Assume that for any compact subset L of Δ, there exists $b, c>0$ such that

$$-Im\,\zeta_1\le b|Im\,\zeta'''|+c|\zeta''|$$

if $P_m(\zeta)=0$ and $Re\,\zeta'''/|Re\,\zeta'''|\in L$. Then the Cauchy problem

$$\begin{cases} P(D)u=0 \\ \left(\dfrac{\partial}{\partial x_1}\right)^j u|_{x_1\to-0}=u_j(x'),\quad j=0,\dots,m-1 \end{cases}$$

admits a hyperfunction solution which contains the same holomorphic parameters.

References

1. J. M. Bony and P. Schapira, *Existence et prolongement des solutions analytiques des systèmes hyperboliques non stricts*, C. R. Acad. Sci. Paris **274** (1972), 86-89.

2. Y. Hamada, J. Leray and C. Wagschal, *Systèmes d'équations aux dérivées partielles à caractéristiques multiples; Problème de Cauchy ramifié; Hyperbolicité partielle*, J. Math. Pures Appl. **55**(1976), 297-352.
3. A. Kaneko, *Estimation of singular spectrum of boundary values for some semi-hyperbolic operators*, J. Fac. Sci. Univ. Tokyo Sec. IA. **27**(1980), 401-461.
4. J. Leray, *Opérateurs partiellement hyperboliques*, C. R. Acad. Sci. Paris **276**(1973), 1685-1687.
5. J. Leray, *Problème de Cauchy I*, Bull. Soc. Math. France **85**(1957), 389-429.

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