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EXPONENTIATION OF HOLOMORPHIC FUNCTIONS

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1. An inequality for the coefficients of composite functions

We fix a positive integer n and let B be the unit ball in $\boldsymbol{C}^n$ and Δ be the unit disk in $\boldsymbol{C}$. For $z, \xi \in \boldsymbol{C}^n$, we let

$$\langle z, \xi\rangle = z_1\bar{\xi}_1 + \cdots + z_n\bar{\xi}_n, \quad \|z\| = \langle z, z\rangle^{1/2}$$

and let $A_q^p(B)$ be the space of all holomorphic functions f in the ball B with

$$\|f\|_{p,q}^p = \int_B |f|^p dv_q < \infty \quad (1 \leq p < \infty, \quad 0 \leq q \leq 1)$$

where for $0 < q \leq 1$

$$dv_q(z) = \frac{\Gamma(n+q)}{\pi^n \Gamma(q)} (1 - \|z\|^2)^{q-1} dv(z)$$

is the probability measure on B and dv_0 denote the unit surface measure on ∂B. We also let $A_{q:s}^p(B)$ be the space of holomorphic functions $f(z) = \sum_{\alpha \geq 0} a_\alpha z^\alpha$ on B with $D^s f \in A_q^p(B)$, where $D^s f(z) = \sum_{\alpha \geq 0} (|\alpha|+1)^s a_\alpha z^\alpha$. We note that when $p=2$, $A_{q:s}^2$ becomes a Hilbert space and

$$\|D^s f\|_{2:q}^2 = \Gamma(q+n) \sum_{\alpha \geq 0} \frac{\alpha!(|\alpha|+1)^{2s}}{\Gamma(n+|\alpha|+q)} |a_\alpha|^2.$$

For other properties of the spaces $A_q^p(B)$ and $A_{q:s}^p(B)$, we refer to [1]. We let $\alpha, \beta, \gamma, \delta$ denote the multi-index with $\alpha! = \alpha_1! \cdots \alpha_n!$, $|\alpha| = \alpha_1 + \cdots + \alpha_n$, $z^\alpha = z_1^{\alpha_1} \cdots z_n^{\alpha_n}$ etc.

Suppose F is a function given by the formal power series

$$F(w) = \sum_{j=0}^{\infty} F_j w^j \quad (F_j \geq 0, \quad w \in \boldsymbol{C}).$$

For any function f defined by the formal expansion

$$f(z) = \sum_{\alpha > 0} a_\alpha z^\alpha \quad (z \in \boldsymbol{C}^n, \quad a_\alpha \in \boldsymbol{C}, \quad \alpha \in \boldsymbol{Z}_+^n),$$

we let $A_\alpha^{(f)}$ be the coefficients of the composite function $F \circ f$, i.e.,

$$\{F \circ f\}(z) = \sum_{j=0}^{\infty} F_j (f(z))^j = A_0^{(f)} + \sum_{\alpha > 0} A_\alpha^{(f)} z^\alpha.$$

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It follows that $A_\alpha^{(f)}$ are functions of a_β with $A_0^{(f)}=F_0$ and in general $A_\alpha^{(f)}=A_\alpha(\{a_\beta\})$, $\beta\leq\alpha$. We will show that $A_\alpha(\{a_\beta\})$ is a linear combination of certain products of powers of a_β. We observe that

$$\sum_{k=0}^{\infty} F_k\Big(\sum_{\alpha>0} a_\alpha z^\alpha\Big)^k=\sum_{\alpha\geq 0}\Big(\sum_{j=0}^{|\alpha|} F_j \sum_{\beta_1+\cdots+\beta_j=\alpha} a_{\beta_1}\cdots a_{\beta_j}\Big) z^\alpha$$

or

(1.1) $$A_\alpha(\{a_\beta\})=\sum_{j=0}^{|\alpha|} F_j\Big(\sum_{\beta_1+\cdots+\beta_j=\alpha} a_{\beta_1}\cdots a_{\beta_j}\Big)=\sum_k A_{\alpha,k}\prod_l a_{\beta l}^{\gamma l},$$

where the products are distinct and $A_{\alpha,k}>0$ when $F_1, \ldots, F_{|\alpha|}>0$. If we write $U_{\alpha,k}(\{\alpha_\beta\})$ for the corresponding products, then using the homogeneity of $U_{\alpha,k}$ in a_β one can easily see that for any coefficients a_α, b_β,

$$U_{\alpha,k}(\{a_\beta b_\beta\})=U_{\alpha,k}(\{a_\beta\})\,U_{\alpha,k}(\{b_\beta\})\ ;$$

for any real number r,

$$|U_{\alpha,k}(\{d_\beta\})|^r=U_{\alpha,k}(\{|d_\beta|^r\})\,;$$

for any $\xi\in\boldsymbol{C}^n$ and $\beta\leq\alpha$,

$$U_{\alpha,k}(\{\xi^\beta\})=\xi^\alpha U_{\alpha,k}(\{1\})=\xi^\alpha,\quad \mathbf{1}=(1,\ldots,1).$$

For $z,\xi\in\boldsymbol{C}^n$ and $0<p<\infty$ we write $z\cdot\xi=(z_1\xi_1,\ldots,z_n\xi_n)$ and $|z|^p=(|z_1|^p,\ldots,|z_n|^p)\in\boldsymbol{R}^n$. A domain D is said to be complete Reinhardt domain if $z\in D$ implies $z\cdot\xi\in D$ for every $\xi\in\bar{\Delta}^n$. If we define

$$D(p)=\{z\in\boldsymbol{C}^n : |z|^{p/2}\in D\},\quad 0<p<\infty$$

then $D(p)$ is a complete Reinhardt domain with $D(2)=D$. We note that if $D=\Delta^n$, then $D(p)=D$ for all $0<p<\infty$. We fix a function ϕ determined by the formal power series

$$\phi(z)=\sum_{\alpha>0} c_\alpha z^\alpha\quad (c_\alpha>0).$$

LEMMA 1.1. *Let α be in $\boldsymbol{Z}_+^n$. Then for $\beta\leq\alpha$ and any p with $1\leq p<\infty$, the Taylor coefficients of the composite function $F\circ f$ satisfy*

$$|A_\alpha(\{a_\beta\})|\leq|A_\alpha(\{c_\beta^{1-p}|a_\beta|^p\})|^{1/p}\cdot|A_\alpha(\{c_\beta\})|^{(p-1)/p}.$$

For $1<p<\infty$, equality holds if $a_\beta=c_\beta\xi^\beta$ for some $\xi\in\boldsymbol{C}^n$. If, in addition, $F_1,\ldots,F_{|\alpha|}>0$, this condition is also necessary.

Proof. We let $d_\alpha=a_\alpha/c_\alpha$ so that $f(z)=\sum_{\alpha>0} c_\alpha d_\alpha z^\alpha$ and

$$A_\alpha(\{a_\beta\})=A_\alpha(\{c_\beta d_\beta\})=\sum_k A_{\alpha,k}U_{\alpha,k}(\{c_\beta d_\beta\}).$$

Using Hölder's inequality with $p'=p/(p-1)$ and the properties of $U_{\alpha,k}$, we obtian

(1.2) $$|A_\alpha(\{a_\beta\})|\leq\sum_k A_{\alpha,k}|U_{\alpha,k}(\{c_\beta d_\beta\})|$$

(1.3) $$\leq\{\sum A_{\alpha,k}U_{\alpha,k}(\{c_\beta|d_\beta|^p\})\}^{1/p}\cdot\{\sum A_{\alpha,k}U_{\alpha,k}(\{c_\beta\})\}^{1/p'}$$

$$=A_\alpha(\{c_\beta|d_\beta|^p\})^{1/p}\cdot A_\alpha(\{c_\beta\})^{1/p'}$$

and the desired inequality follows. When $a_\beta=c_\beta\xi^\beta$, $\beta\leq\alpha$, $\xi\in\boldsymbol{C}^n$ we have

$$A_\alpha(\{c_\beta\xi^\beta\})=\sum\xi^\alpha A_{\alpha,k}U_{\alpha,k}(\{c_\beta\})=\xi^\alpha A_\alpha(\{c_\beta\}).$$

Thus

$$|A_\alpha(\{a_\beta\})|=|\xi^\alpha|\cdot A_\alpha(\{c_\beta\})^{1/p}\cdot A_\alpha(\{c_\beta\})^{1/p'}$$
$$=A_\alpha(\{c_\beta|\xi^\beta|^p\})^{1/p}\cdot A_\alpha(\{c_\beta\})^{1/p'}$$

and the equality holds in this case. Conversely, assume the equality holds. Writing $c_\beta d_\beta=\eta_\beta c_\beta|d_\beta|$, $\eta_\beta\in\boldsymbol{C}$, $|\eta_\beta|=1$, we have

$$U_{\alpha,k}(\{c_\beta d_\beta\})=U_{\alpha,k}(\{\eta_\beta\})U_{\alpha,k}(\{c_\beta|d_\beta|\}).$$

It follows that (1.2) becomes equality if and only if $U_{\alpha,k}(\{\eta_\beta\})$ is independent of k and (1.3) becomes equality if and only if $A_{\alpha,k}U_{\alpha,k}(\{c_\beta|d_\beta|^p\})=c(\alpha)A_{\alpha,k}U_{\alpha,k}(\{c_\beta\})$ for some constant $c(\alpha)>0$ and every k. Since $A_{\alpha,k}>0$ when $F_1,\ldots,F_{|\alpha|}>0$, we conclude that

$$U_{\alpha,k}(\{d_\beta\})=c(\alpha)^{1/p}e^{i\theta}\quad(0\leq\theta<2\pi)$$

for every k. In particular, all monomials of the form $\prod_l d_{\beta l}^{\eta_l}$ are equal. Hence $d_\beta=\xi^\beta$ for every $\beta\leq\alpha$ where $\xi=(d_1,\ldots,d_n)\in\boldsymbol{C}^n$. This completes the proof.

Remark. When $p=2$ this result appears in [3] with different proof, while $n=1$ and $1\leq p<\infty$ the result appears in [5]. This remark applys to the theorem below as well.

We fix a complete Reinhardt domain D such that $\phi(z\cdot\bar z)=\sum c_\alpha|z|^{2\alpha}<\infty$, $z\in D$ and $\phi(z\cdot\bar z)=\infty$, $z\in\partial D$. Then we see that $\phi(|z|^p)<\infty$, $z\in D(p)$, $1\leq p<\infty$ and we have

Theorem 1.2. *If for some $p\geq1$ the Taylor coefficients of f and F satisfy*

(1.4) $$\sum_{\alpha>0}c_\alpha^{1-p}|a_\alpha|^p=\sigma<\infty,\quad\sum_{k>0}F_k\sigma^k<\infty,$$

then

(1.5) $$\sum_{\alpha\geq0}A_\alpha(\{c_\beta\})^{1-p}|A_\alpha(\{a_\beta\})|^p\leq F(\sum_{\alpha>0}c_\alpha^{1-p}|a_\alpha|^p).$$

For $1<p<\infty$, equality holds if $a_\alpha=c_\alpha\xi^\alpha$ for all $\alpha>0$ and for some $\xi\in D(p)$. This condition is also necessary if $F_m>0$, $m=1,2,\ldots$

Proof. By Lemma 1.1, we have

(1.6) $$A_\alpha(\{c_\beta\})^{1-p}|A_\alpha(\{a_\beta\})|^p\leq A_\alpha(\{c_\beta^{1-p}|a_\beta|^p\}.$$

Fix an integer N and sum over all α with $|\alpha|\leq N$ to obtain

$$\sum_{|\alpha|\leq N}A_\alpha(\{c_\beta\})^{1-p}|A_\alpha(\{a_\beta\})|^p\leq\sum_{|\alpha|\leq N}A_\alpha(\{c_\beta^{1-p}|a_\beta|^p\}).$$

By (1.4), the function $g(z)=\sum_{\alpha>0} c_\alpha^{1-p}|a_\alpha|^p z^\alpha$ is holomorphic in Δ^n, while $F(w)$ is holomorphic in the disk $|w|<\sigma$. Hence $(F\circ g)(z)=\sum A_\alpha(\{c_\beta^{1-p}|a_\beta|^p\})z^\alpha$ is holomorphic in Δ^n. Moreover,

$$\sum_{|\alpha|\le N} A_\alpha(\{c_\beta^{1-p}|a_\beta|^p\})\le\sum_{k=0}^{\infty}F_k(g(\mathbf{1}))^k=F(\sigma),$$

where $\mathbf{1}=(1,\ldots,1)$. Note that $g(\mathbf{1})$ and $F(\sigma)$ are both well defined. Hence by Abel's theorem

$$\sum_{\alpha\ge 0} A_\alpha(\{c_\beta^{1-p}|a_\beta|^p\})=\lim_{r\to 1^-}F(g(\boldsymbol{r}))=F(\sigma),$$

where $\boldsymbol{r}=(r_1,\ldots,r_n)\in\boldsymbol{R}^n$. Letting $N\to\infty$, we obtain the result. For $1<p<\infty$, equality holds if and only if equality holds in (1.6) for every $\alpha>0$, which is true if $a_\alpha=c_\alpha\xi^\alpha$ for some $\xi\in\boldsymbol{C}^n$. When $F_m>0$, $m=1,2,\ldots$, $a_\alpha=c_\alpha\xi^\alpha(\alpha)$ by Lemma 1.1, we need to show that ξ is independent of α. Without loss of generality, we assume $a_\alpha\ne 0$ for some α and we let $a_\gamma=c_\gamma\xi^\gamma(\gamma)$ and $a_\delta=c_\delta\xi^\delta(\delta)$ and let $\beta\le\alpha$ with $\gamma\le\beta$, $\delta\le\alpha$. Then $\xi(\gamma)=\xi(\beta)$ and $\xi(\delta)=\xi(\alpha)$. But since also $\beta\le\alpha$, we have $a_\beta=c_\beta\xi^\beta$ with $\xi=\xi(\alpha)$. Hence $\xi(\alpha)=\xi(\beta)$ and thus $\xi(\gamma)=\xi(\delta)$. Moreover, in this case $\phi(|\xi|^p)<\infty$ and hence $\xi\in D(p)$.

2. Application for the exponential function

In this section, we define certain space of sequences and study exponentiation of holomorphic functions. For $0<p<\infty$ we define l_ϕ^p as the space of all sequences $\{a_\alpha\}$, $\alpha\in\boldsymbol{Z}_+^n$ with

$$\|\{a_\alpha\}\|_{l_\phi^p}=\{\sum_{\alpha>0}c_\alpha^{1-p}|a_\alpha|^p\}^{1/p}<\infty.$$

We also define $l_\phi^p(D(p'))$ $(p'=p/(p-1),\ 1<p<\infty)$ as the space of all holomorphic functions $f(z)=\sum_{\alpha>0}a_\alpha z^\alpha(z\in D(p'))$ with

$$\|f\|_{p,\phi}=\|\{a_\alpha\}\|_{l_\phi^p}<\infty.$$

Then the mapping $\Lambda: l_\phi^p\to l_\phi^p(D(p'))$ defined as

$$\Lambda(\{a_\alpha\})=\sum_{\alpha>0}a_\alpha z^\alpha$$

is an isometry, since if $\{a_\alpha\}\in l_\phi^p$ then

$$\sum_{\alpha>0}|a_\alpha z^\alpha|\le\{\sum_{\alpha>0}c_\alpha^{1-p}|a_\alpha|^p\}^{1/p}\{\sum_{\alpha>0}c_\alpha|z^\alpha|^{p'}\}^{1/p'}$$
$$=\|\{a_\alpha\}\|_{l_\phi^p}\{\phi(|z|^{p'})\}^{1/p'}<\infty$$

for all $z\in D(p')$. Hence $f\in l_\phi^p(D(p'))$ and since $\|\{a_\alpha\}\|_{l_\phi^p}=\|f\|_{p,\phi}$, $f\in l_\varphi^p(D(p'))$ implies $\{a_\alpha\}\in l_\varphi^p$, completing the proof.

We choose $F(\lambda)=e^\lambda$, $\lambda\in\boldsymbol{C}$ and $\phi(z)=-q\log(1-\langle z,\mathbf{1}\rangle)$. $z\in B$, $q>0$. In this case we find that $c_\alpha=q\Gamma(|\alpha|)/\alpha!$ and that the coefficients of

$F\circ\phi$ are $\Gamma(q+|\alpha|)/\Gamma(\alpha)\alpha!$. If $f(z)=\sum_{\alpha>0} a_\alpha z^\alpha \in l_\phi^p(B(p'))$ then f satisfies (1.4) and (1.5) becomes

(2.1) $$\sum_{\alpha\geq 0}(\Gamma(q+|\alpha|)/\Gamma(q)\alpha!)^{1-p}|A_\alpha^{(f)}|^p \leq \exp\Big(\sum_{\alpha>0}(q\Gamma(|\alpha|)/\alpha!)^{1-p}|a_\alpha|^p\Big).$$

Equality holds if and only if $a_\alpha=q\Gamma(|\alpha|)/\alpha!\bar{\xi}^\alpha$ for some $\xi\in B(p)$, i.e., if and only if $f(z)=-q\log(1-\langle z,\xi\rangle)$ for some $\xi\in B(p)$. In the terminology introduced in the beginning of this section, we have just proved the following:

THEOREM 2.1. *Let* $1<p<\infty$ *and* $p'=p/(p-1)$. *Then* $e^f\in l_{e^\phi}^p(B(p'))$ *whenever* $f\in l_\phi^p(B(p'))$ *with*

$$\|e^f\|_{p,e^\phi}^p\leq\exp(\|f\|_{p,\phi}^p).$$

Equality holds if and only if $f(z)=\phi(z\cdot\bar{\xi})$ *for some* $\xi\in B(p)$.

As a special case, if $p=2$, we get the following result of Burbea [3].

COROLLARY 2.2. *Let* k *be any positive integer with* $2k-n\geq 0$ *and let* $q\geq n$. *Then for any* $f\in A_{2k-n:k}^2$ *with* $f(0)=0$, *there exists a constant* c *independent of* f *such that*

$$\|e^f\|_{q-n}^2\leq\exp(c\|D^kf\|_{2k-n}^2).$$

In particular, $e^f\in A_{q-n}^2$.

Proof. We have, by Sterling's formula, for $f\in A_{2k-n:k}^2$ with $f(0)=0$,

$$\|D^kf\|_{2k-n}^2=\Gamma(2k)\sum_{\alpha>0}\frac{\alpha!}{\Gamma(|\alpha|+2k)}(|\alpha|+1)^{2k}|a_\alpha|^2 \sim c\sum_{\alpha>0}\frac{\alpha!}{\Gamma(|\alpha|)}|a_\alpha|^2$$

while $\|e^f\|_{q-n}^2=\Gamma(q)\sum_{\alpha\geq 0}\frac{\alpha!}{\Gamma(q+|\alpha|)}|A_\alpha^{(f)}|^2$. This completes the proof.

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