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BRAUER GROUP OVER A KRULL DOMAIN

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Let R be a Krull domain with field of fractions K. By $\mathrm{Br}(R)$ we denote the Brauer group of R. Studying the Kernel of the homomorphism $\mathrm{Br}(R) \to \mathrm{Br}(K)$, Orzech defined Brauer groups $\mathrm{Br}(M)$ for different categories M of R-modules [4].

In this paper we show that an algebra A in $\mathrm{Br}(D)$ is a maximal order in $A \otimes K$ and that the map $\mathrm{Br}(D) \to \mathrm{Br}(K)$ is one to one.

We note here few conventions. All rings are Krull domains and all modules will be unitary. By Z we donote the set of height one prime ideals of a Krull domain.

0. Preliminaries

We first recall the following definitions and basic properties taken from [4].

(1) An R-module M is divisorial if it is torsion free and in $K \otimes M$ the equality $M = \bigcap_{P \in Z} M_P$ holds.

(2) An R-module M is an R-lattice if M is torsion free of finite rank and there exists an R-module F of finite type such that $M \subset F \subset M \otimes K$.

Let D be the category of divisorial R-lattices. For M and N in D, we view $M \otimes N$ as a subset of $(M \otimes_R K) \otimes_K (N \otimes_R K)$ and define

$$M \perp N = \bigcap_{P \in Z} (M \otimes N)_P$$

Let $\mathrm{Az}(D)$ be the set of isomorphism classes of central R-algebras A which are in D as R-modules, and for which the following natural map $\eta_A : A \perp A^0 \to \mathrm{End}_R(A)$ induced by the map $A \otimes A^0 \to \mathrm{End}_R(A)$ is an isomorphism. We note that R-algebra A is in $\mathrm{Az}(D)$ if and only if A is a divisorial R-lattice and A_P is an R_P-Azumaya algebra (i. e. A_P is a central separable R_P-algebra) for all p in Z.

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We define an equivalence relation $\sim$ on $\mathrm{Az}(D)$ by

$$A \sim B \text{ if } A \perp \mathrm{End}_R(M) \simeq B \perp \mathrm{End}_R(N)$$

for some M and N in D. Let $\mathrm{Br}(D)$ denote the set of equivalence classes of $\mathrm{Az}(D)$ and let $[A]$ denote the class of A. Then $\mathrm{Br}(D)$ is an abelian group under the operation $[A][B]=[A \perp B]$. The identity element in this group is given by $[\mathrm{End}_R(E)]$ for some $E \in D$ and $[A]^{-1}=[A^0]$.

Since every faithfully projective R-module is in D, there is a group homomorphism $\mathrm{Br}(R) \to \mathrm{Br}(D)$. For any R-algebra A and any prime ideal p of R $A \otimes_R K \simeq A_P \otimes_{R_p} K$, we have the induced group homomorphism $\mathrm{Br}(D) \to \mathrm{Br}(K)$. In [4] Orzech proved that the kernel of the map $\mathrm{Br}(R) \to \mathrm{Br}(K)$ is exactly the kernel of the map $\mathrm{Br}(R) \to \mathrm{Br}(D)$.

1. Main Theorem

Let R be a regular domain and let R-algebra A be an Azumaya algebra. Then it is well known that A is a maximal R-order in $A \otimes K$ [3]. Similarly the following holds:

PROPOSITION 1. *Let R-algebra A be a divisorial R-lattice such that A_P is a central separable R_P-algebra for every $p \in Z$ (i.e. $[A] \in Br(D(R))$). Then A is a maximal R-order in $K \otimes A$.*

Proof. Let B be the integral closure of R in A. Since B contains a K-basis of $K \otimes A$ which is in A, B is an R-order in $K \otimes A$. For each $p \in Z$, A_p is an (maximal) R-order in $K \otimes A_p$ by proposition 6.18 [3] and hence $B_p = A_p$. By proposition 6.11 [3] B^{**} is an R-order in $K \otimes A$. From the following canonical inclusions

$$B \subset B^{**} \subset A$$

and $B_p = A_p$, we have $i_p : B^{**}{}_p \simeq A_p$. Since B^{**} and A are divisorial (equivalently reflexive) R-modules, $i : B^{**} \to A$ is an isomorphism by Lemma 1.1 [4] and hence A is an R-order in $K \otimes A$. Since A_p is a maximal R-order in $K \otimes A_p$ for each $p \in Z$, by proposition 1.3 [1], A is a maximal R-order in $A \otimes K$.

THEORM 2. *The map $Br(D) \to Br(K)$ is a monomorphism.*

Proof. Let $[A]$ be in $\mathrm{Br}(D)$ which becomes trivial in $\mathrm{Br}(K)$. Then there is a finite dimensional vector space V over K such that $A \otimes K \simeq \mathrm{Hom}_K(V, V)$. By Proposition 1, A is a maximal R-order in the central simple K-algebra $\mathrm{Hom}_K(V, V)$. By Proposition 1.7 [1], $A \simeq \mathrm{End}_R(E)$ for some divisorial R-lattice E. By the definition of $\mathrm{Br}(D)$, $[\mathrm{End}_R(E)] = [A]$ is the identity element in $\mathrm{Br}(D)$ and hence the map is a monomorphism.

COROLLARY. *$Br(D)$ is a torsion group.*

References

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