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STABLE SPLITTINGS OF BG FOR GROUPS WITH PERIODIC COHOMOLOGY AND UNIVERSAL STABLE ELEMENTS

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1. Introduction

This paper deals with the classifying spaces of finite groups. To any finite group G we associate a space BG with the property that $\pi_1(BG)=G$, $\pi_i(BG)=0$ for $i>1$. BG is called the classifying space of G.

Consider the problem of finding a stable splitting

$$BG \simeq X_1 \vee X_2 \vee \cdots \vee X_n$$

localized at p. Ideally the X_i's are indecomposable, thus displaying the homotopy type of BG in the simplest terms. Such a decomposition naturally splits $H^*(BG)$.

The main purpose of this paper is to give the classification theorem in stable homotopy theory for groups with periodic cohomology i.e. cyclic Sylow p-subgroups for p an odd prime and to calculate some universal stable elements.

In this paper, all cohomology groups are with Z/p-coefficients and p is an odd prime.

REMARK. A classification theorem for groups with periodic cohomology at $p=2$ has been given by Martino and Priddy [6].

2. Classification Theorem

Let H be a subgroup of a finite group G. Let $i_{H,G}$ denote the inclusion $H \to G$ and $t_{H,G}$ denote the transfer, which is a stable map. Since $H^*(BG)=H^*(G)$ (group cohomology), we frequently write i^* in place of $(Bi)^*$. Recall [1, Chap. 12].

PROPOSITION 2.1. *On ordinary cohomology,* $(it)^*$ *is multiplication by* $[G:H]$.

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Corollary 2.2. *If $p \nmid [G : H]$, then BG is a stable summand of BH.*

Definition 2.3. Let H be a subgroup of a finite group G. Then $x \in H^*(BH)$ is called stable if for all $g \in G$,

$$\begin{array}{ccc} & H^*(BH) & \\ i^* \swarrow & & \searrow j^* \\ H^*(B(H\cap Hg)) & \xrightarrow{c_g^*} & H^*(B(Hg^{-1}\cap H)) \end{array}$$

commute.

Proposition 2.4 [10]. *Suppose the Sylow p–subgroup Gp of a finite group G is abelian. Let Φ_p be the group of automorphisms of Gp induced by inner automorphisms of G. Then an element $\alpha \in H^*(Gp)$ is stable if and only if it is fixed under the action of Φ_p on $H^*(Gp)$.*

The group Φ_p is, of course, isomorphic to $N(Gp)/C(Gp)$ where N and C denote the normalizer and centralizer, respectively.

Proposition 2.5 [1, Chap. 12]. (1) *If α is in the image of i^*, then α is stable.*

(2) *Let Gp be a Sylow p–subgroup of a finite group G. Then the restriction map induces an isomorphism onto the stable elements.*

Whenever a group Γ acts on a module M, M^Γ denotes the module of Γ–invariants.

Then by Proposition 2.4 and 2.5 we can get the following.

Lemma 2.6. *If a Sylow p–subgroup Gp of a finite group G is cyclic (i.e., $Gp \cong Z/p^n$), then*

$$H^*(BG) = H^*(BGp)^{N/C}.$$

It is well known that

$$H^*(BZ/p^n) = Z/p[y] \otimes E[x]$$

where $\dim x = 1$ and $\dim y = 2$.

Lemma 2.7. *If a Sylow p–sugroup Gp of a finite group G is cyclic, then*

$$H^*(BGp)^{N/C} = Z/p[y^r] \otimes E[xy^{r-1}]$$

where $r = |N/C|$.

Proof. $N(Z/p^n)/C(Z/p^n)$ is isomorphic to a subgroup of $\mathrm{Aut}(Z/p^n)$ and $\mathrm{Aut}(Z/p^n)$ is cyclic. Thus $N/C \cong Z/r$ where $r = |N/C|$.

Let α be a generator of $Z/r \subset (Z/p^n)^X$. Since

$$H^1(BZ/p^n) = \mathrm{Hom}(Z/p^n, Z/p)$$

and

$$H^*(BZ/p^n) = Z/p[y] \otimes E[x]$$

where dim $x=1$ and $\beta x = y$ ($\beta \in H^2(BZ/p^n)$ is the Bockstein),

$$\alpha^*(\langle x \rangle) \cong \langle \alpha x \rangle$$
$$\alpha^*(\langle y \rangle) \cong \langle \alpha y \rangle.$$

Thus

$$\alpha^*(y^r) = \alpha^r y^r = y^r$$
$$\alpha^*(xy^{r-1}) = \alpha x \alpha^{r-1} y^{r-1} = xy^{r-1}$$

Therefore,

$$H^*(BZ/p^n)^{N/C} \supset Z/p[y^r] \otimes E[xy^{r-1}].$$

The rest of the proof is obvious.

Proposition 2.8[2]. *There is a p-local stable homotopy equivalence*

$$BZ/p^n \simeq X_1 \vee \cdots \vee X_{p-1}$$

where all X_i's are simply connected spaces and $H^*(X_i) = Z/p$ only in dimension of the forms $2k(p-1)+2i$ and $2k(p-1)+2i-1$, $H^*(X_i)=0$ otherwise.

Theorem 2.9 (Classification theorem). (1) *If a Sylow p-subgroup Gp of a finite group G is cyclic, then there is a p-local stable homotopy equivalence*

$$BG \simeq \bigvee_{j=1}^{p-1/r} X_{jr}$$

where $r = |N/C|$.

(2) *All such decompositions arise for some finite group G.*

Proof. (1) $H^*(BG) = H^*(BGp)^{N/C}$ (by Lemma 2.6)

$= Z/p[y^r] \otimes E[xy^{r-1}]$ (by Lemma 2.7)

By Proposition 2.8

$$\bigvee_{j=1}^{p-1/r} X_{jr} \hookrightarrow \bigvee_{i=1}^{p-1} X_i \simeq BG_p \longrightarrow BG$$

induces an isomorphism in cohomology and is therefore a p-local stable homotopy equivalence. Then we prove the result.

(2) Let $G = Z/p^n \times Z/r$ where $Z/r \subset GL_1(Z/p^n) = Z/p^n - p^{n-1}$.

Then using the Serre spectral sequence,

$$H^*(G) = H^*(Z/p^n)^{Z/r}$$

so,

$$BG \simeq \bigvee_{j=1}^{p-1/r} X_{jr}.$$

THEOREM 2.10. *If a Sylow p-subgroup Gp of a finite group G is cyclic, then there is an epimorphism*

$$\phi : N_G(Gp) \to Z/p^n \rtimes Z/r$$

where r is the order of N/C, which induces an isomorphism ϕ^*

Proof. By proof of Lemma 2.7,

$$N/C \cong Z/r$$

where $r=|N/C|$. Since Gp is abelian and contained in $C(Gp)$, there is an epimorphism $N(Gp)/Gp \to Z/r$. But since the order of $N(Gp)/Gp$ is prime to p, an extension $Gp \rightarrowtail N_G(Gp) \twoheadrightarrow N_G(Gp)/Gp$ splits

Thus

$$N_G(Gp) \cong Gp \rtimes N_G(Gp)/Gp$$
$$\cong Z/p^n \rtimes N_G(Gp)/Gp.$$

Therefore, we can get an epimorphism

$$\phi : N_G(Gp) \to Z/p^n \rtimes Z/r.$$

Now we show that ϕ^* is an isomorphism.

$$H^*(N(Gp)) \cong H^*(Gp)^{N(Gp)/Gp}$$
$$H^*(Gp \rtimes Z/r) \cong H^*(Gp)^{N/C}$$

Since Gp and $C(Gp)$ act trivially on $H^*(Gp)$,

$$H^*(N(Gp)) \cong H^*(Gp)^N \cong H^*(Gp \rtimes Z/r).$$

3. Universal stable elements

DEFINITION 3.1. Let P be a finite *p-group*. Define $\mathfrak{I}(P)$ the intersection of the images of monomorpisms $i^* : H^*(BG) \to H^*(BP)$ for all G with $Gp \cong P$. $\mathfrak{I}(P)$ is called the set of universal stable elements of $H^*(BP)$.

By Proposition 2.5, $\mathfrak{I}(Gp)$ is the intersections of all stable elements over G having Sylow p-subgroup. Of course, $\mathfrak{I}(Gp)$ forms a ring.

REMARK. $H^*(B\Sigma_p) = \mathfrak{I}(N/p)$ since $N(Z/p) = Z/p-1 \rtimes Z/p^{-1}$. We also have

$$X_{p-1} \simeq B(Z/p \rtimes Z/p-1) \simeq B\Sigma_p.$$

THEOREM 3.2. *Let Gp be an cyclic Sylow p-subgroup of a finite group G. Then*

(1) $\mathfrak{I}(Z/p^n) = Z/p[y^{p-1}] \otimes E[xy^{p-2}]$

(2) $H^*(B(Z/p^n \rtimes Z/p-1)) = \mathfrak{I}(Z/p^n)$

Proof. (1) Let a map $N(Gp) \to \mathrm{Aut}(Gp)$ be given by $r \to Cr$ where Cr is the conjugation map $x \longmapsto r^{-1}xr$.
Then the induced map $N/C \to \mathrm{Aut}(Gp)$ is a monomorphism. But

Thus $$\mathrm{Aut}(Gp) \cong Z/p^n - p^{n-1} \cong Z/p^{n-1} \oplus Z/p-1$$
$$\begin{aligned} H^k(BZ/p^n)^{N/C} &\supset H^k(BZ/p^n)^{\mathrm{Aut}(Gp)} \\ &\cong (Z/p)^{\mathrm{Aut}(Gp)} \\ &\cong (Z/p)^{Z/p-1} \end{aligned}$$

Since Z/p^{n-1} acts trivially on Z/p.

Now let $\mathcal{G}$ be the set of all finite groups G with cyclic Sylow p-subgroups $Gp \cong Z/p^n$. Therefore,

$$\begin{aligned} \mathfrak{J}(Z/p^n) &= \bigcap_{Z/p^n \in \mathcal{G}} \mathrm{Im}\{H^*(BZ/p^n) \overset{i^*}{\longleftarrow} H^*(BG)\} \\ &\cong \bigcap i^*H^*(BG) \\ &= \bigcap H^*(BZ/p^n)^{N/C} \\ &\supset H^*(BZ/p^n)^{Z/p-1} \end{aligned}$$

On the other hand, from the Serre spectral sequence, we can get
$$H^*(B(Z/p^n \rtimes Z/p-1) = H^*(BZ/p^n)^{Z/p-1}$$

So $\mathfrak{J}(Z/p^n) \subset H^*(BZ/p^n)^{Z/p-1}$. Thus
$$(Z/p^n) = H^*(BZ/p^n)^{Z/p-1},$$

Therefore, using the similar method of Lemma 2.7, we prove the result.

(2) It is obvious.

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