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GAUSSIAN MEASURES ON LORENTZ SEQUENCE SPACES

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1. Introduction

In 1935 A. Kolmogorov[6] introduced the concept of a characteristic functional $\hat{\mu}$ of the measure μ on the Banach space X as a function from X^* into $\boldsymbol{C}$ defined by $\hat{\mu}(x^*)=\int_X \exp\{i\langle x^*, x\rangle\} d\mu(x)$, $x^*\in X^*$.

In 1951 M. Fréchet[5] defined a Gaussian distribution on an infinite dimensional space as follows: A random variable x with values in a Banach space X is called Gaussian if for each $x^*\in X^*$, a random variable $\langle x^*, x\rangle$ is a scalar Gaussian.

In 1953 E. Mourier[11] defined the expectation of a Banach space valued random variable using the Pettis integral as follows: Let x be a random variable with values in a Banach space X such that for each $x^*\in X^*$, $x^*(x)$ has a finite expected value. If there exists an element a of X such that for each $x^*\in X^*$, $x^*(a)=\mathrm{E}\langle x^*, x\rangle$ then a is called the expectation of x and a is denoted by $\mathrm{E}x$. She showed the following:

(a) Let x be a Gaussian random variable with values in a separable Hilbert space H. Then the expectation of x exists and $\mathrm{E}\|x\|^2$ is finite.

(b) The characteristic functional $\hat{\mu}$ of a Gaussian measure μ on a separable Hilbert space H is given by $\hat{\mu}(h)=\exp\left\{i\mathrm{E}\langle h, x\rangle - \frac{1}{2}\mathrm{E}\langle h, x-\mathrm{E}x\rangle^2\right\}$, $h\in H$. $(*)$ Conversely, if $\mathrm{E}\|x\|^2$ is finite then $(*)$ is the characteristic functional of a Gaussian measure on H.

Let $\boldsymbol{R}^N$ denote the linear space of all real numerical sequences,

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equipped with the product topology. Then the space of all continuous linear functionals defined on $\boldsymbol{R}^N$ is the space $\boldsymbol{R}_0^N$ consisting of all finitely supported sequences.

Let Σ be the σ-algebra generated by sets of the form $\{x\in R^N : (x_1, x_2, \cdots x_n)\in B^n\}$, where B^n is a Borel set in $\boldsymbol{R}^n$. A subset E of $\boldsymbol{R}^N$ is called measurable if $E\in\Sigma$.

A distribution or probability measure μ on $\boldsymbol{R}^N$ is a nonnegative σ-additive measure defined on Σ such that $\mu(\boldsymbol{R}^N)=1$. The probability measure μ on $\boldsymbol{R}^N$ is called Gaussian if for each $f\in\boldsymbol{R}_0^N$, the random variable $f(x)$ is a scalar Gaussian.

Let X be a measurable subspace of $\boldsymbol{R}^N$. A subset A of X is called measurable in X if there exists a measurable subset B of $\boldsymbol{R}^N$ such that $A=B\cap X$. By definition, if $A\subset X$ is measurable in X then A is measurable in $\boldsymbol{R}^N$. Hence if μ is a distribution on $\boldsymbol{R}^N$ with $\mu(X)=1$ then μ is also a distribution on X. Conversely, for each distribution μ on X there exists a distribution $\tilde{\mu}$ on $\boldsymbol{R}^N$ defined as $\tilde{\mu}(E)=\mu(E\cap X)$ for $E\in\Sigma$. It follows from $R_0^N\subset X^*$ that the probability measure μ on X is Gaussian if and only if the corresponding distribution $\tilde{\mu}$ on $\boldsymbol{R}^N$ is Gaussian. Therefore we would obtain all possible Gaussian distributions on X if we could find all possible Gaussian distributions on R^N that are concentrated on X.

N. N. Vakhania [12] extended the result of E. Mourier [11], which is a description of the general form of the characteristic functionals of all Gaussian distributions on a separable Hilbert space, as follows:

(a) In order that a Gaussian distribution on $\boldsymbol{R}^N$ with expectation $\{a_k\}$ and covariance matrix $[S_{ij}]$ be concentrated on l_p, $1\leq p<\infty$, it is necessary and sufficient that $\{a_k\}\in l_p$ and $\{S_{kk}\}\in l_{\frac{p}{2}}$.

(b) Let μ be a Gaussian distribution on $\boldsymbol{R}^N$ with $(\{a_k\}, [S_{ij}])$ as above. If $\{a_k\}\in C_0$ and $\sum_{k=1}^{\infty} e^{-\frac{\varepsilon}{S_{kk}}}<\infty$ for any $\varepsilon>0$ then μ is concentrated on C_0. Conversely, if $\mu(C_0)=1$ then $\{a_k\}\in C_0$, and if the covariance matrix is diagonal then $\sum_{k=1}^{\infty} e^{-\frac{\varepsilon}{S_{kk}}}<\infty$ for any $\varepsilon>0$.

(c) Let μ be a Gaussian distribution on $\boldsymbol{R}^N$ with $(\{a_k\}, [S_{ij}])$ as above. If $\{a_k\} \in l_\infty$ and $\sum_{k=1}^{\infty} e^{-\frac{r}{S_{kk}}} < \infty$ for some $r > 0$ then μ is concentrated on l_∞. Conversely, if $\mu(l_\infty) = 1$ then $\{a_k\} \in l_\infty$, and if the covariance matrix is diagonal then $\sum_{k=1}^{\infty} e^{-\frac{r}{S_{kk}}} < \infty$ for some $r > 0$.

Given a non-increasing sequence of positive numbers $W = \{W_n\}_{n=1}^{\infty}$, and $1 \leq p < \infty$, the Lorentz sequence space $d(W, p)$ is defined to be the collection of all sequences $\{x_n\}_{n=1}^{\infty}$ for which $\sum_{n=1}^{\infty} (x_n^*)^p W_n < \infty$, where $\{x_n^*\}$ is the decreasing rearrangement of $\{|x_n|\}$. The Lorentz sequence space $l_{p,q}$, $1 \leq q \leq p < \infty$, is equal to $d(W, q)$ with $W_n = n^{\frac{q}{p}-1}$. If $p = q$ then $l_{p,q} = l_p$.

By using N. N. Vakhania's proof, we intend to generalize the above results as follows: Let $\{W_n\}_{n=1}^{\infty} \notin l_1$. In order that a Gaussian distribution on $\boldsymbol{R}^N$ with expectation $\{a_k\}$ and covariance matrix $[S_{ij}]$ be concentrated on $d(W, p)$, $1 \leq p < \infty$ (and hence to be Gaussian on $d(W, p)$), it is necessary and sufficient that $\{a_k\} \in d(W, p)$ and $\{S_{kk}\} \in d\left(W, \frac{p}{2}\right)$.

2. Definitions and notation

We present some of the definitions and notation to be used.

Notation.

(1) $\{x_n^*\}$ denotes the decreasing rearrangement of $\{|x_n|\}$.

(2) $W = \{W_n\}_{n=1}^{\infty}$ denotes a non-increasing sequence of positive numbers.

(3) For $1 \leq p < \infty$, $l_p(W)$ denotes the set of all sequences $x = \{x_n\}_{n=1}^{\infty}$ for which $\|x\|_{l_p(W)} < \infty$, where $\|x\|_{l_p(W)} = (\sum_{j=1}^{\infty} |x_j|^p W_j)^{\frac{1}{p}}$.

(4) ρ denotes a permutation of the positive integers.

(5) W_ρ denotes a sequence $\{W_{\rho(n)}\}_{n=1}^{\infty}$.

(6) $\boldsymbol{N}$ denotes the set of all positive integers.

(7) $\tilde{\eta} = \{(n_1, \cdots, n_k) \mid n_j \in \boldsymbol{N},\ k \in \boldsymbol{N},\ n_i \neq n_j \text{ if } i \neq j\}$.

(8) For each $\vec{n} = (n_1, \cdots, n_k) \in \tilde{\eta}$, let $\rho_{\vec{n}}$ be some permutation such that $\rho_{\vec{n}}(n_i) = i$ for $i = 1, 2, \cdots, k$. Then let η be the (countable)

collection of these ρ_n's.

Let $1\leq p<\infty$ and let $W=\{W_n\}_{n=1}^{\infty}$ be a non-increasing sequence of positive numbers. A Lorentz sequence space $d(W,p)$ is the Banach space of all sequences $x=\{x_n\}_{n=1}^{\infty}$ for which $\|x\|_{d(W,p)}<\infty$, where $\|x\|_{d(W,p)}=(\sum_{n=1}^{\infty}(x_n^*)^p X_n)^{\frac{1}{p}}$. The classical Lorentz sequence space $l_{p,q}$, $1\leq q\leq p<\infty$, is $d(W,q)$, where $W_n=n^{\frac{q}{p}-1}$.

We define the space $\bigcap_{\rho} l_p(W_\rho)$ as follows: The space $\bigcap_{\rho} l_p(W_\rho)$ consists of all sequences $x=\{x_n\}_{n=1}^{\infty}$ with finite $\|x\|_{\bigcap_\rho l_p(W_\rho)}$, where $\|x\|_{\bigcap_\rho l_p(W_\rho)}=\sup_\rho\{(\sum_{n=1}^{\infty}|x_n|^p W_{\rho(n)})^{\frac{1}{p}}\}$. It is easy to see that $\bigcap_\rho l_p(W_\rho)$ is a Banach space. (The proof is standard so it is omitted).

3. Results

First we see that if X is a Banach space in $\boldsymbol{R}^N$ and X has a basis $\{x_k\}_{k=1}^{\infty}$ then X is measurable in $\boldsymbol{R}^N$ in a reasonable sense.

Let $\{x_k^*\}$ be the sequence of biorthogonal functionals associated with the basis $\{x_k\}$. Since $\boldsymbol{R}^N$ has the product topology, a function j from X into $\boldsymbol{R}^N$ defined by $j(x)=\{x_k^*(x)\}_{k=1}^{\infty}$ for all $x\in X$ is continuous. We know that $\{x_k^*\}$ is a weak* basis for X^*. Hence the set $\{\sum_{k=1}^{n}a_k x_k^* : a_k\in Q\}$ is weak* dense in X^*. Let $œ=\{\sum_{k=1}^{n}a_k x_k^* : a_k\in Q,\ \|\sum_{k=1}^{n}a_k x_k^*\|\sim 1\}$ and $B_X=\{x\in X : \|x\|\leq 1\}$. Since $\|x\|=\sup_{\|x^*\|=1}|\langle x^*, x\rangle|$, we have $j(B_X)=\bigcap_{œ}\{\{b_k\}\in\boldsymbol{R}^N : |\sum_{k=1}^{n}a_k b_k|\leq 1\}$. Therefore $j(B_X)$ is a countable intersection of measurable sets in $\boldsymbol{R}^N$ and so $j(B_X)$ is measurable in R^N. In this sense, B_X is measurable in $\boldsymbol{R}^N$. Since $X=\bigcup_{n=1}^{\infty}\{x : \|x\|\leq n\}$, X is measurable in $\boldsymbol{R}^N$. Hence $l_p(W)$, $\bigcap_\rho l_p(W_\rho)$ and $d(W,p)$ are measurable in $\boldsymbol{R}^N$.

Now we state the following two simple lemmas.

Lemma 1. *Let $\xi_1(\omega)$, $\xi_2(\omega)$, $\cdots$ be an arbitrary sequence of random variables and $\{\alpha_n\}_{n=1}^{\infty}$ a sequence of positive numbers that increase monotonically to infinity. If $P\{\omega : \xi_n(\omega)\geq\alpha_n\}\geq constant>0$,* *then the sequence* $\xi_1(\omega)$, $\xi_2(\omega)$, $\cdots$ *cannot be bounded with probability* 1. *In particular, it cannot be almost surely convergent.*

(The proof can be found in [12]).

LEMMA 2. *Let* $\xi(\omega)$ *be a Gaussian random variable on the probability space*$(\Omega, \mathscr{B}, P)$ *with expectation* m *and variance* σ^2. *Then for each* $t \geq 0$ *the following relation is true.* $\int_\Omega |\xi(\omega)-m|^t P(d\omega) = C(t)\sigma^t$, *where* $C(t) = 2^{\frac{t}{2}}\Pi^{-\frac{1}{2}}\Gamma\left(\frac{1+t}{2}\right)$ *and* Γ *is Euler's gamma function.*

(The proof can be found in [12]).

Using these two lemmas, we give necessary and sufficient conditions for a Gaussian measure μ on $\boldsymbol{R}^N$ to be supported in $l_p(W)$, $1 \leq p < \infty$.

PROPOSITION 1. *If the Gaussian distribution* μ *is concentrated on* $l_p(W)$, $1 \leq p < \infty$, *then* $\{a_j\} \in l_p(W)$, *where* $a_j = \int x_j \mu(dx)$.

Proof. Assume the contrary, that is, $\{a_j\} \notin l_p(W)$. Then we can find an element f in the dual space $(l_p(W))^*$ so that the series $\sum_{k=1}^{\infty} a_k f_k$ diverges. Changing the signs of the corresponding coordinates of the element f, if necessary, we can take all the terms of this series to be positive, and therefore the sequence of partial sums of this series increases monotonically to ∞. Consider the functional $f^{(n)}(x) = \sum_{k=1}^{n} f_k x_k$. This functional is linear and continuous and therefore has a Gaussian distribution. Since $\int \sum_{k=1}^{n} f_k x_k \mu(dx) = \sum_{k=1}^{n} f_k a_k$, by the normality of the distribution, the following equalities hold.

$$\mu\{x : \sum_{k=1}^{n} f_k x_k \geq \sum_{k=1}^{n} f_k a_k\} = \frac{1}{2} \quad (n=1, 2, \cdots).$$

By lemma 1, the series $\sum_{k=1}^{\infty} f_k x_k$ should diverge on a set with positive probability. But $x \in l_p(W)$ almost surely and $f \in (l_p(W))^*$ implies that $\mu\{x : |\sum_{k=1}^{\infty} f_k x_k| < \infty\} = 1$. Therefore $\mu\{x : |\sum_{k=1}^{\infty} f_k x_k| = \infty\} = 0$ and so $\mu\{x : \sum_{k=1}^{\infty} f_k x_k = \infty\} = 0$. This contradiction proves the proposition.

PROPOSITIPN 2. *If the Gaussian distribution μ is concentrated on $l_p(W)$, $1 \leq p < \infty$, then $\{S_{kk}\} \in l_{\frac{p}{2}}(W)$, where $S_{kk} = \int (x_k - a_k)^2 \mu(dx)$.*

Proof. Denote by A_n $(n = 1, 2, \cdots)$ the following events. $\{x : \sum_{k=1}^{n} |x_k - a_k|^p W_k \geq \alpha(1-\beta) S^{(n)}\}$, where $S^{(n)} = \sum_{j=1}^{n} S_{jj}^{\frac{p}{2}} W_j$, $\alpha > 0$, $0 < \beta < 1$.

The proof of the proposition will be complete if we show that (1) $\mu(A_n) \geq \text{constant} > 0$ $(n = 1, 2, \cdots)$ for some values of α and β satisfying the condition $\alpha(1-\beta) > 0$. In fact, if the sequence $\{S^{(n)}\}$ diverges, then the lemma 1 tells us that the series $\sum_{k=1}^{\infty} |x_k - a_k|^p W_k$ diverges on a set of positive probability because of (1). But this is impossible because $\{a_k\} \in l_p(W)$ almost surely and $\{x_k\} \in l_p(W)$ almost surely. Therefore we shall show that inequality (1) is satisfied by a suitable choice of the numbers α and β. Since $\{x : |\sum_{k=1}^{n} |x_k - a_k|^p W_k - \alpha S^{(n)}| \leq \alpha\beta S^{(n)}\} \subseteq A_n$, we have $\mu(A_n) \geq \mu\{x : |\sum_{k=1}^{n} |x_k - a_k|^p W_k - \alpha S^{(n)}| \leq \alpha\beta S^{(n)}\}$. Using Chebyshev's inequality, we obtain

$$\mu(A_n) \geq 1 - \frac{1}{\alpha^2 \beta^2 S^{(n)2}} \Big[\sum_{j,k=1}^{n} \int |x_j - a_j|^p W_j |x_k - a_k|^p W_k \mu(dx) - 2\alpha S^{(n)} \sum_{k=1}^{n} \int |x_k - a_k|^p W_k \mu(dx) + \alpha^2 S^{(n)2} \Big].$$

Using Hölder's inequality and applying the lemma 2, we obtain

$$\begin{aligned} &\sum_{j,k=1}^{n} \int |x_j - a_j|^p W_j |x_k - a_k|^p W_k \mu(dx) \\ &\leq \sum_{j,k=1}^{n} W_j W_k \Big(\int |x_j - a_j|^{2p} \mu(dx)\Big)^{\frac{1}{2}} \Big(\int |x_k - a_k|^{2p} \mu(dx)\Big)^{\frac{1}{2}} \\ &= \sum_{j,k=1}^{n} W_j W_k (C(2p) S_{jj}^{p})^{\frac{1}{2}} (C(2p) S_{kk}^{p})^{\frac{1}{2}} = C(2p)(S^{(n)})^2. \end{aligned}$$

Again using the lemma 2, we have

$$\sum_{k=1}^{n} \int |x_k - a_k|^p W_k \mu(dx) = C(p) S^{(n)}.$$

Therefore $\mu(A_n) \geq 1 - \frac{1}{\alpha^2 \beta^2} [C(2p) - 2\alpha C(p) + \alpha^2]$.

Now it is sufficient to show the existence of the values $\alpha>0$ and $0<\beta<1$ for which the following inequality is satisfied. $\frac{1}{\alpha^2\beta^2}[C(2p)-2\alpha C(p)+\alpha^2]<1$. Let α satisfy the condition $C(2p)-2\alpha C(p)<0$. Then $\alpha^{-2}[C(2p)-2\alpha C(p)+\alpha^2]<1$ and we can choose a positive number β so that $\alpha^{-2}[C(2p)-2\alpha C(p)+\alpha^2]<\beta^2<1$. This ends the proof of the proposition.

PROPOSITION 3. *Let μ be a Gaussian distribution on $\boldsymbol{R}^N$ with expectation $\{a_k\}$ and covariance matrix $[S_{ij}]$. If $\{a_k\}\in l_p(W)$ and $\{S_{kk}\}\in l_{\frac{p}{2}}(W)$ then μ is concentrated on $l_p(W)$.*

Proof. From the inequality $|x_j|^p\leq 2^p(|x_j-a_j|^p+|a_j|^p)$, it follows that

$$\int\sum_{j=1}^{\infty}|x_j|^p W_j\mu(dx)\leq 2^p\int\sum_{j=1}^{\infty}|x_j-a_j|^p W_j\mu(dx)+2^p\int\sum_{j=1}^{\infty}|a_j|^p W_j\mu(dx).$$

Since $\{a_k\}\in l_p(W)$, $\sum_{j=1}^{\infty}|a_j|^p W_j<\infty$.

By the lemma 2 and the monotone convergence theorem, we have $\int\sum_{j=1}^{\infty}|x_j-a_j|^p W_j\mu(dx)=C(p)\|S_{kk}\|_{l_{\frac{p}{2}}(W)}^{\frac{p}{2}}$.

Therefore $\int\sum_{j=1}^{\infty}|x_j|^p W_j\mu(dx)<\infty$.

Note that $l_p(W)=\bigcup_{n=1}^{\infty}\{x:\sum_{j=1}^{\infty}|x_j|^p W_j<n^p\}$. By using the continuity of the measure and Chebychev's inequality we obtain

$$\mu((l_p(W))^c)\leq\lim_n\frac{1}{n^p}\int(\sum_{j=1}^{\infty}|x_j|^p W_j)\mu(dx).$$

Hence $\mu((l_p(W))^c)=0$ and so $\mu(l_p(W))=1$.

LEMMA 3. *Let $\tilde{\eta}=\{(n_1,\cdots n_k)\,|\,n_j\in N,\ k\in N,\ n_i\neq n_j \text{ if } i\neq j\}$. For each $\vec{n}=(n_1,\cdots,n_k)\in\tilde{\eta}$, let $\rho_{\vec{n}}$ be some permutation such that $\rho_{\vec{n}}(n_i)=i$ for $i=1,2,\cdots,k$. Then let η be the countable collection of these $\rho_{\vec{n}}$'s. Suppose that $\{W_k\}_{k=1}^{\infty}\notin l_1$. If $\|x\|_{l_p(W_\rho)}<\infty$ for every $\rho\in\eta$ then $x\in C_0$.*

Proof. Assume that $x\notin C_0$. Then there is $\delta>0$ such that for any $i>0$, there exists $n_i\geq i$ with $|x_{n_i}|\geq\delta$. For each k, let ρ_k be some permutation such that $\rho_k(n_i)=i$, $i=1,2,\cdots k$. Then we have

$$\|x\|_{l_p(W_{\rho_k})}=(\sum_{i=1}^{\infty}|x_i|^p W_{\rho_k(i)})^{\frac{1}{p}}\geq(\sum_{i=1}^{k}|x_{n_i}|^p W_i)^{\frac{1}{p}}\geq\delta(\sum_{i=1}^{k} W_i)^{\frac{1}{p}}.$$

Since $\{W_k\}\notin l_1$, $\lim \|x\|_{l_p(W_{\rho_k})}=\infty$ as $k\to\infty$. The proof follows from this contradiction.

REMARKS.

1. If $x\in C_0$ then there exists a permutation ρ such that
$$(\sum_{n=1}^{\infty}(x_n^*)^p W_n)^{\frac{1}{p}}=(\sum_{n=1}^{\infty}|x_n|^p W_{\rho(n)})^{\frac{1}{p}}.$$
2. If $x\in C_0$ then $\sup\limits_{\rho}\ \{(\sum_{n=1}^{\infty}|x_n|^p W_{\rho(n)})^{\frac{1}{p}}\}=(\sum_{n=1}^{\infty}|x_n^*|^p W_n)^{\frac{1}{p}}$.
3. If $\{W_k\}\notin l_1$ then $\bigcap\limits_{\rho} l_p(W_\rho)=d(W,p)$.
4. Since $x_1^*(W_1)^{\frac{1}{p}}\leq(\sum_{n=1}^{\infty}(x_n^*)^p W_n)^{\frac{1}{p}}\leq x_1^*(\sum_{n=1}^{\infty}W_n)^{\frac{1}{p}}$, if $\{W_k\}\in l_1$ then it follows that $d(W,p)=l_\infty$.

LEMMA 4. *If* $\{W_k\}\notin l_1$ *then* $\sup\limits_{\rho}\{\|x\|_{l_p(W_\rho)}\}=\sup\limits_{\rho\in\eta}\{\|x\|_{l_p(W_\rho)}\}$.

Proof. Since $\tilde{\eta}$ is countable, η is a countable subset of the set of all permutations of N. Therefore
$$\sup_{\rho}\{\|x\|_{l_p(W_\rho)}\}\geq\sup_{\rho\in\eta}\{\|x\|_{l_p(W_\rho)}\}\quad (1)$$
So if $\sup\limits_{\rho\in\eta}\{\|x\|_{l_p(W_\rho)}\}=\infty$ then $\sup\limits_{\rho}\{\|x\|_{l_p(W_\rho)}\}=\sup\limits_{\rho\in\eta}\{\|x\|_{l_p(W_\rho)}\}$.

Suppose that $\sup\limits_{\rho\in\eta}\{\|x\|_{l_p(W_\rho)}\}<\infty$. Then, by lemma 3, $x\in C_0$. Therefore, from remark 1, it follows that there exists a permutation ρ such that $x_i^*=|x_{\rho(i)}|$, $i=1,2,\cdots$. For each k there is a permutation σ_k which belongs to η such that $\sigma_k(\rho_{(j)})=j$ for $j=1,2,\cdots k$. Then $\sup\limits_{\rho\in\eta}\{\|x\|_{l_p(W_\rho)}\}\geq\|x\|_{l_p(W_{\sigma_k})}\geq(\sum_{j=1}^{k}x_{\rho(j)}|^p W_j)^{\frac{1}{p}}$.

But $\lim\limits_{k\to\infty}(\sum_{j=1}^{k}|x_{\rho(j)}|^p W_j)^{\frac{1}{p}}=(\sum_{j=1}^{\infty}(x_j^*)^p W_j)^{\frac{1}{p}}$ and it follows from remark 2 that $(\sum_{j=1}^{\infty}(x_j^*)^p W_j)^{\frac{1}{p}}=\sup\limits_{\rho}\{\|x\|_{l_p(W_\rho)}\}$.

Therefore we have $\sup\limits_{\rho\in\eta}\{\|x\|_{l_p(W_\rho)}\}\geq\sup\limits_{\rho}\{\|x\|_{l_p(W_\rho)}\}$ (2).

From (1) and (2) we get the result.

THEOREM. *Suppose that* $\{W_k\}\notin l_1$. *In order for the Gaussian* *measure on* $\boldsymbol{R}^N$ *with expectation* $\{a_k\}$ *and covariance matrix* $[S_{ij}]$ *to be concentrated on* $d(W,p)$, $1\leq p<\infty$, *it is necessary and sufficient that* $\{a_k\}\in d(W,p)$ *and* $\{S_{kk}\}\in d\left(W,\frac{p}{2}\right)$.

Proof. Necessity.

Suppose that $\mu(d(W,p))=1$. Note that $\bigcap_\rho l_p(W_\rho)=d(W,p)$ by remark 3. By definition, if $x\in\bigcap_\rho l_p(W_\rho)$ then $x\in l_p(W_\rho)$ for every permutation ρ. Hence we have $\mu(\bigcap_\rho l_p(W_\rho))\leq\mu(l_p(W_\rho))$ for every permutation ρ, and so $\mu(l_p(W_\rho))=1$ for every permutation ρ. Now by proposition 1, we have $\{a_k\}\in l_p(W_\rho)$ for every permutation ρ and hence $\{a_k\}\in C_0$ by lemma 3. Therefore, by remark 1 and 2, there exists a permutation ρ such that $(\sum_{j=1}^\infty|a_j|^pW_{\rho(j)})^{\frac{1}{p}}=(\sum_{j=1}^\infty(a_j^*)^p W_j)^{\frac{1}{p}}=\sup_\rho\{(\sum_{j=1}^\infty|a_j|^pW_{\rho(j)})^{\frac{1}{p}}\}$ and so $\{a_k\}\in d(W,p)$. It follows from proposition 2 that $\{S_{kk}\}\in l_{\frac{p}{2}}(W_\rho)$ for every permutation ρ. Again, by the same reason as above, we see that $\{S_{kk}\}\in d\left(W,\frac{p}{2}\right)$. This completes the proof of necessity.

Sufficiency.

Suppose that $\{a_k\}\in d(W,p)$ and $\{S_{kk}\}\in d\left(W,\frac{p}{2}\right)$. Then, by remark 3, the series $(\sum_{j=1}^\infty|a_j|^pW_{\rho(j)})^{\frac{1}{p}}$ is finite and the series $(\sum_{k=1}^\infty|S_{kk}|^{\frac{p}{2}} W_{\rho(k)})^{\frac{2}{p}}$ is finite for every permutation ρ. Hence by proposition 3, we have $\mu(l_p(W_\rho))=1$ for every permutation ρ.

Now, it follows from the proof of lemma 4 that there is a permutation σ which belongs to η so that $\sup_{\rho\in\eta}\{\|x\|_{l_p(W_\rho)}\}=\|x\|_{l_p(W_\sigma)}$ and so $\bigcap_{\rho\in\eta} l_p(W_\rho)=\{x:\sup_{\rho\in\eta}\{\|x\|_{l_p(W_\rho)}\}<\infty\}$. Hence it follows from lemma 4 and remark 3 that $\bigcap_{\rho\in\eta} l_p(W_\rho)=\bigcap_\rho l_p(W_\rho)=d(W,p)$.

Now we show that $\mu(\bigcap_{\rho\in\eta} l_p(W_\rho))=1$ and this proves the sufficiency. Since $\mu(\bigcup_{\rho\in\eta}[l_p(W_\rho)]^c)\leq\sum_\eta\mu([l_p(W_\rho)]^c)$ and $\mu([l_p(W_\rho)]^c)=0$ for every $\rho\in\eta$, we have $\mu(\bigcup_{\rho\in\eta}[l_p(W_\rho)]^c)=0$. By taking the complement, we get $\mu(\bigcap_{\rho\in\eta} l_p(W_\rho))=1$.

Corollary. *In order for the Gaussian measure on* $\boldsymbol{R}^N$ *with expectation* $\{a_k\}$ *and covariance matrix* $[S_{ij}]$ *to be concentrated on* $l_{p,q}$, $1\leq q\leq p<\infty$, *it is necessary and sufficient that* $\{a_k\}\in l_{p,q}$ *and* $\{S_{kk}\}\in l_{\frac{p}{2},\frac{q}{2}}$.

Proof. The Lorentz sequence space $l_{p,q}$, $1\leq q\leq p<\infty$, is equal to $d(W,q)$, where $W_n=n^{\frac{q}{p}-1}$. It follows from $1\leq q\leq p<\infty$ that $\{W_n\}\notin l_1$. Hence the result follows from theorem.

By remark 4, if $\{W_k\}\in l_1$ then $d(W,p)=l_\infty$. But the necessary and sufficient conditions for a Gaussian measure μ to be concentrated on l_∞ are known. Therefore, from our theorem we found, for every Lorentz sequence space $d(W,p)$, necessary and sufficient conditions for a Gaussian measure μ to be concentrated on $d(W,p)$.

It follows from [12] that if $\{a_k\}\in\boldsymbol{R}^N$ and $[S_{ij}]$ is a symmetric and nonnegative definite matrix then a function $\hat{\mu}$ from $\boldsymbol{R}_0^N$ into $\boldsymbol{C}$ defined by $\hat{\mu}(f)=\exp\{i\sum f_k a_k-\frac{1}{2}\sum_{i,j}S_{ij}f_if_j\}$, $f\in\boldsymbol{R}_0^N$ is a characteristic functional of a Gaussian measure μ on $\boldsymbol{R}^N$ with expectation $\{a_k\}$ and covariance matrix $[S_{ij}]$. We know from our theorem if $\{a_k\}\in d(W,p)$ and $\{S_{kk}\}\in d\left(W,\frac{p}{2}\right)$ then $\mu(d(W,p))=1$ under the condition that $W\notin l_1$. Therefore μ is a Gaussian measure on $d(W,p)$, $W\notin l_1$, with expectation $\{a_k\}$ and covariance matrix $[S_{ij}]$. Hence $\hat{\mu}(f)=\exp\{i\sum_k f_k a_k-\frac{1}{2}\sum_{i,j}S_{ij}f_if_j\}$, $f\in d(W,p)^*$, $W\notin l_1$ is a characteristic functional of a Gaussian measure μ on $d(W,p)$, $W\notin l_1$, with expectation $\{a_k\}$ and covariance matrix $[S_{ij}]$. Hence we have the following: A function $\hat{\mu}: d(W,p)^*\to\boldsymbol{C}$ is the characteristic functional of a Gaussian measure μ on $d(W,p)$, $W\notin l_1$ if and only if it is of the form $\hat{\mu}(f)=\exp\{i\sum_k f_k a_k-\frac{1}{2}\sum_{i,j}S_{ij}f_if_j\}$, $f\in d(W,P)^*$, where $\{a_k\}\in d(W,p)$, $W\notin l_1$ and $\{S_{kk}\}\in d\left(W,\frac{p}{2}\right)$, $W\notin l_1$.

After I had proved theorem, I found that our theorem is a special case of Chobanjan and Tarieladze's theorem which states that for a Banach space X which has an unconditional basis $\{x_k\}_{k=1}^{\infty}$ and has cotype p for some $p<\infty$, a functional $\hat{\mu}: X^* \to C$ is a characteristic functional of a Gaussian measure μ on X if and only if it admits the representation $\hat{\mu}(x^*)=\exp\{i<x^*, a>-\frac{1}{2}\langle Sx^*, x^*\rangle\}$, where $a \in X$ and $S: X^* \to X$ is a nonnegative symmetric bounded linear operator such that the series $\sum_{k=1}^{\infty}\langle Sx_k^*, x_k^*\rangle^{\frac{1}{2}} x_k$, where $\{x_k^*\}$ is the sequence of biorthogonal functionals associated with the basis $\{x_k\}$, is convergent in X (cf. [3]). But our proof is quite different from Chobanjan and Tarieladze's. Our method in proving theorem is rather direct, but Chobanjan and Tarieladze's proof depends crucially on Maurey and Pisier's result which says that if a Banach space X has cotype p for some $p<\infty$ then every bounded linear operator from l_∞ into X is p-summing.

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