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A NOTE ON REAL HYPERSURFACES OF A COMPLEX SPACE FORM

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Introduction

In 1973 Takagi [7] classified homogeneous hypersurfaces of a complex projective space P^nC by proving that all of them could be divided into six types, and he [8], [9] showed also if a real hypersurface M has two or three distinct constant principal curvatures, then M is congruent to one of the homogeneous hypersurfaces of type A_1, A_2 or B among these ones. This result is generalized by many others [3], [4] and [6].

On the other hand, many subjects for real hypersurfaces of a complex hyperbolic space H^nC were investigated from different points of view ([1], [5] etc.), one of which, done by Chen, Ludden and Montiel [1] asserted that a real hypersurface M of H^nC is of *cyclic-parallel* if and only if the structure tensor J induced on M and the shape operator A derived from the unit normal commute each other, that is, $JA=AJ$. In particular, real hypersurfaces of H^nC, which are said to type A, similar to those of type A_1 and A_2 of P^nC were treated by Montiel and Romero [5].

Recently one of the present authors [2] asserted that a real hypersurface of a complex space form $M^n(c)$, $c\neq 0$, is of cyclic parallel if and only if $AJ=JA$ and he showed also a complete and connected cyclic-parallel real hypersurface of $M^n(c)$ is congruent to type A_1, A_2 or Aaccording as $c>0$ or $c<0$.

A real hypersurface of a complex space form $M^n(c)$ is said to be *covariantly cyclic constant* if the cyclic sum of covariant derivative of the second fundamental form is constant. The purpose of the present paper is to extend theorem 3 and 4 in [2] when the hypersurfaces are of covariantly cyclic constant, that is,

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THEOREM. *A real hypersurface of a complex space form* $M^n(c)$, $c \neq 0$, *is of covariantly cyclic constant if and only if* $AJ=JA$, *and a complete and connected covariantly cyclic constant real hypersurface of* $M^n(c)$ *is congruent to type* A_1, A_2 *or* A *according as* $c>0$ *or* $c<0$.

1. Preliminaries

We begin with recalling fundamental properties on real hypersurfaces of a complex space form. Let $M^n(c)$ be a real $2n$-dimensional complex space form endowed with a metric tensor G of a constant holomorphic sectional curvature c and a parallel almost complex structure F, and be covered by a system of coordinate neighborhoods $\{U; x^h\}$. Let M be a real hypersurface of $M^n(c)$ covered by a system of coordinate neighborhoods $\{V; y^h\}$ and immersed isometrically in $M^n(c)$ by the immersion $i: M \to M^n(c)$. Throughout the present paper the following convention on the range of indices are used, unless otherwise stated:

$$A, B, \cdots = 1, 2, \cdots, 2n\ ;\ i, j, \cdots = 1, 2, \cdots, 2n-1.$$

The summation convention will be used with respect to those system of indices. When the argument is local, M need not be distinguished from $i(M)$. Thus, for simplicity, a point p in M may be identifed with the point $i(p)$ and a tangent vector X at p may also be identified with the tangent vector $i_*(X)$ at $i(p)$ via the differential i_* of i. We represent the immersion i locally by $x^A = x^A(y^h)$ and $B_j = (B_j^A)$ are also $(2n-1)$-linearly independent local tangent vectors of M, where $B_j^A = \partial_j x^A$ and $\partial_j = \partial/\partial y^j$. A unit normal C to M may then be chosen. The induced Riemannian metric g with components g_{ji} on M is given by $g_{ji} = G(B_j, B_i)$ because the immersion is isometric.

For the unit normal C to M, the following representations are obtained in each coordinate neighborhood :

$$FB_i = J_i^h B_h + P_i C, \quad FC = -P^i B_i, \tag{1.1}$$

Where we have put $J_{ji} = G(FB_j, B_i)$ and $P_i = G(FB_i, C)$, P^h being components of a vector field P associated with P_i and $J_{ji} = J_j^r g_{ri}$. By the properties of the almost Hermitian structure F, it is clear that J_{ji} is skew-symmetric. A tensor field of type (1, 1) with components J_i^h will be denoted by J. By the properties of the almost complex structure F, the following relations are then given :

$$J_i^r J_r^h = -\delta_i^h + P_i P^h, \quad P^r J_r^h = 0, \quad P_r J_i^r = 0, \quad P_i P^i = 1, \quad g_{st} J_j^s J_i^t = g_{ji} - P_j P_i,$$

that is, the aggregate (J, g, P) defines an almost contact metric structure. Denoting by ∇_j the operator of Van der Waerden-Bortolotti covariant differentiation formed with g_{ji}, equations of the Gauss and Wiengarten for M are respectively obtained:

$$\nabla_j B_i = h_{ji} C, \quad \nabla_j C = -h_j^r B_r, \tag{1.2}$$

where h_{ji} are components of a second fundamental form $\sigma, A = (h_j^h)$ which is related by $h_{ji} = h_j^r g_{ri}$ being the shape operator derived from C. We notice here that h_{ji} is symmetric. By means of (1.1) and (1.2) the covariant derivatives of structure tensors are yielded:

$$\nabla_i J_{ih} = -h_{ji} P_h + h_{jh} P_i, \quad \nabla_j P_i = -h_{jr} J_i^r. \tag{1.3}$$

Since $M^n(c)$ is of constant holomorphic sectional curvature c, the Gauss and Codazzi equations are respectively given:

$$R_{kjih} = \frac{c}{4}(g_{kh} g_{ji} - g_{jh} g_{ki} + J_{kh} J_{ji} - J_{jh} J_{ki} - 2J_{kj} J_{ih}) + h_{kh} h_{ji} - h_{jh} h_{ki}, \tag{1.4}$$

$$\nabla_k h_{ji} - \nabla_j h_{ki} = \frac{c}{4} A_{kji}, \quad A_{kji} = P_k J_{ji} - P_j J_{ki} - 2P_i J_{kj}, \tag{1.5}$$

where R_{kjih} are the components of the Riemannian curvature tensor R of M. Let S_{ji} be the components of the Ricci tensor S of M, then the Gauss equation implies

$$S_{ji} = \frac{c}{4}\{(2n+1) g_{ji} - 3P_j P_i\} + h h_{ji} - h_{ji}^2, \tag{1.6}$$

where h denotes the trace of the shape operator A and $h_{ji}^2 = h_{jr} h_i^r$.

2. Proof of the Theorem

Let M be a real hypersurface of a complex space form $M^n(c)$. The hypersurface M is called *covariantly cyclic constant* if the cyclic sum of $\nabla\sigma$ is constant, namely

$$\nabla_m (\nabla_k h_{ji} + \nabla_j h_{ik} + \nabla_i h_{kj}) = 0. \tag{2.1}$$

Throughout the present paper we only consider the case where the holomorphic sectional curvature c is not zero.

From now on we suppose that M is of covariantly cyclic constant. Then we have from (1.5)

(2.2) $$3\nabla_m\nabla_k h_{ji}=\frac{c}{4}(\nabla_m A_{kji}+\nabla_m A_{kij}).$$

By the second equation of (1.5) and (1.3), it follows that

(2.3) $$\nabla_m\nabla_k h_{ij}=\frac{c}{4}\{(\nabla_m P_j)J_{ik}+(\nabla_m P_i)J_{jk}-h_{mi}P_jP_k-h_{mj}P_kP_i+2h_{mk}P_jP_i\}.$$

If we substitute (1.4) and (2.3) into the Ricci formula for h_{ji}, which is given by

$$\nabla_m\nabla_k h_{ji}-\nabla_k\nabla_m h_{ji}=-R_{mkjr}h_i^r-R_{mkir}h_j^r,$$

then we have

(2.4) $$h_{ik}^2h_{mj}-h_{im}^2h_{kj}+h_{jk}^2h_{im}-h_{jm}^2h_{ik}$$
$$=\frac{c}{4}\{h_{mi}(g_{kj}-P_kP_j)-h_{ki}(g_{mj}-P_mP_j)+h_{jm}(g_{ki}-P_kP_i)$$
$$-h_{jk}(g_{mi}-P_mP_i)+J_{jk}(\nabla_mP_i+\nabla_iP_m)-J_{jm}(\nabla_kP_i+\nabla_iP_k)$$
$$+J_{ik}(\nabla_mP_j+\nabla_jP_m)-J_{im}(\nabla_kP_j+\nabla_jP_k)+2J_{mk}(\nabla_jP_i+\nabla_iP_j)\},$$

where we have used the second equation of (1.3).

If we contract this to the indices j and i,

(2.5) $$hh_{ji}^2=\left\{h_2-\frac{c}{2}(n+1)\right\}h_{ji}+ch_{rs}J_j^rJ_i^s$$
$$+\frac{c}{2}\{(h_{jr}P^r)P_i+(h_{ir}P^r)P_j\}+\frac{c}{4}h(g_{ji}-P_jP_i),$$

where $h_2=h_{ji}h^{ji}$, which yields

(2.6) $$hh_{jr}^2P^r=\left(h_2-\frac{c}{2}n\right)h_{jr}P^r+\frac{c}{2}\alpha P_j,$$

where we have defined $\alpha=h_{rs}P^rP^s$. Thus it follows that

(2.7) $$h\beta=\left\{h_2-\frac{c}{2}(n-1)\right\}\alpha,\quad \beta=h_{ji}^2P^jP^i.$$

On the other hand, if we transvect (2.4) with J^{ik} and $P^jP^iP^k$ respectively and make use of the properties of the almost contact metric structure (J,g,P), then we can see that

(2.8) $$J^{sr}(h_{sm}h_{jr}^2+h_{js}h_{mr}^2)$$
$$=\frac{c}{4}(2n+1)(\nabla_jP_m+\nabla_mP_j)-\frac{c}{4}\{(P^r\nabla_rP_j)P_m+(P^r\nabla_rP_m)P_j\},$$

(2.9) $$\alpha h_{mr}^2P^r=\beta h_{mr}P^r.$$

Combining (2.6) and (2.7) with (2.9), it follows that $\alpha(h_{jr}P^r-\alpha P_j)=0$ and hence $\alpha(\beta-\alpha^2)=0$.

Let M_1 be a set consisting of points of M at which the function $\beta-\alpha^2$ does not vanish and suppose that M_1 is not empty. Then M_1

is an open submanifold of M. We have $\alpha=0$ and thus $\beta h_{mr}P^r=0$ on M_1 because of (2.9). By transvecting $h^m_s P^s$, it follows that $\beta^2=0$ and hence β vanishes on M_1. Thus it is a contradiction. Accordingly we have $\beta=\alpha^2$ on M, which means that P is the principal curvature vector corresponding to α, that is,

(2.10) $$h_{jr}P^r=\alpha P_j.$$

Applying P^m to (2.8) and summing up m, we obtain

(2.11) $$P^r\nabla_r P_j=0$$

because of the fact that $c\neq 0$. Now we will prove that the principal curvature α is constant. Differentiating (2.10) covariantly along M, we find

(2.12) $$(\nabla_k h_{jr})P^r+h_{jr}\nabla_k P^r=\alpha_k P_j+\alpha\nabla_k P_j,$$

where $\alpha_k=\nabla_k\alpha$.

By means of (1.5) and the properties of the almost contact metric structure (J, g, P), we then have

(2.13) $$-\frac{c}{2}J_{kj}+2h_{kr}h_{js}J^{rs}=\alpha_k P_j-\alpha_j P_k-\alpha(h_{kr}J^r_j-h_{jr}J^r_k).$$

Applying P^k to above equation and summing up to k, we can see $\alpha_j=BP_j$ for some function B on M. Differentiating this covariantly, we find $\nabla_j\alpha_i=(\nabla_j B)P_i-Bh_{jr}J^r_i$ by means of (1.3) and hence $B(h_{jr}J^r_i-h_{ir}J^r_j)=0$. Let M_0 be a set consisting of points at which the function B dose not vanish and suppose that M_0 is not empty. Then M_0 is an open submanifold of M and we have $h_{jr}J^r_i-h_{ir}J^r_j=0$ on M_0. Thus (2.13) is reduced to $-\frac{c}{2}J_{kj}+2h_{kr}h_{js}J^{rs}=0$, which together with (2.8) and (2.11) gives $h_{jr}J^r_i+h_{ir}J^r_j=0$. Consequently $h_{jr}J^r_i=0$, which produces a contradiction. Thus, by the definition of B, we see that α is constant everywhere.

Transvecting (2.4) with P^jP^m, we can easily verify that

(2.14) $$\alpha h^2_{ik}-\alpha^2 h_{ik}-\frac{c}{4}\alpha(g_{ki}-P_kP_i)=0.$$

If $\alpha\neq 0$, then we have $h_2-\alpha h-\frac{c}{2}(n-1)=0$, which enables us to know $h^r_j J^k_r=J^r_j h^k_r$. But if $\alpha=0$, (2.13) implies $h_{kr}h_{js}J^{rs}=\frac{c}{4}J_{kj}$, which means $h_{jr}J^r_i+h_{ir}J^r_j=0$ by the same discussion as above. Consequently we see in any case that $h^r_j J^k_r=J^r_j h^k_r$. Combining this with Theorem 3 and 4 in [2], the theorem is completely proved.

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