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THE TRANSFORMATION GROUPS AND THE ISOMETRY GROUPS

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1. Introduction

Methods of Riemannian geometry has played an important role in the study of compact transformation groups. Every effective action of a compact Lie group on a differential manifold leaves a Riemannian metric invariant and the study of such actions reduces to the one involving the group of isometries of a Riemannian metric on the manifold which is, a priori, a Lie group under the compact open topology. Once an action of a compact Lie group is given an invariant metric is easily constructed by the averaging method and the Lie group is naturally imbedded in the group of isometries as a Lie subgroup. But usually this invariant metric has more symmetries than those given by the original action. Therefore the first question one may ask is when one can find a Riemannian metric so that the given action coincides with the action of the full group of isometries.

This seems to be a difficult question to answer which depends very much on the orbit structure and the group itself. In this paper we give a sufficient condition that a subgroup action of a compact Lie group has an invariant metric which is not invariant under the full action of the group and figure out some aspects of the action and the orbit structure regarding the invariant Riemannian metric. In fact, according to our results, this is possible if there is a larger transformation group, containing the oringnal action and either having larger orbit somewhere or having exactly the same orbit structure but with an orbit on which a Riemannian

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metric is ivariant under the orginal action of the group and not under that of the larger one.

Recently R. Saerens and W. Zame showed that every compact Lie group can be realized as the full group of isometries of Riemannian metric. [SZ] This answers a question closely related to ours but the situation turns out to be quite different in the two problems.

2. Preliminaries

Everything in this paper will be assumed connected.

Let M be a differentiable manifold. Given a Riemannian metric g on M, it is well known that the group of isometries of g with the compact open topology is a Lie group[MS] and that the topology is equivalent to the C^∞ topology and hence is equivalent to every topology —C^k topology for example—between them. [EK]

To every compact transformation group G can be associated a Riemannian metric invariant under the action by averaging an arbitrary Riemannian metric g on M, i. e.

$$g_{\mathrm{inv}}(X, Y)=\int_{\alpha\in G} g(\alpha_* X, \alpha_* Y)$$

Thus G can be imbedded into $\mathrm{Isom}(g_{\mathrm{inv}})$, the group of isometries of g_{inv}. [B]

For a Remannian metric g on a manifold $\mathrm{Isom}(g)$ has a semi-continuity property, namely

(2. 1) There is a C^2 neighborhood U in the space of Riemannian metrics on M such that, if $g_1 \in U$, there is a smooth map $\eta : M \to M$ with $\eta \cdot \mathrm{Isom}(g_1) \cdot \eta^{-1} \subset \mathrm{Isom}(g)$. [E, GK, K]

Given an action of a compact Lie group G on M, at each point $x \in M$, there is a slice S_x which is an open submanifold of M transversal to the orbit $G(x)$ of x such that

(i) S_x is invariant under the isotropy subgroup G_x, and

(ii) $G(S_x)=G\times_{G_x} S_x$, i. e. a linear tube which is a twisted product. [BP]

It is well known that for a compact Lie transformation groups on a compact manifold M there are only finitely many distinct orbit types and among them is the unique principal orbit type, that is, the orbit type with highest dimensional orbits. Moreover the orbits of principal type form a dense subset and the union of them is an open submanifold of M so that its complement has codimension at least 1. [B, M]

3. Main theorems and proofs

THEOREM 1. *Let G and H be compact Lie groups acting effectively on a compact differentiable manifold M. Let g be a Riemannian metric on M which is invariant under the action of G. If $H(x)\not\subset G(x)$ for some x in M, (where $G(x)$ is the orbit of x under the action of G) there is an arbitrarily small C^2 deformation of g which is still invariant under the action of G but not under that of H.*

Proof. First we find a one parameter subgroup K of H such that $K(x)\not\subset G(x)$, Then the closure ClK is a compact abelian subgroup of H and therefore it is isomorphic to a torus. Hence we can find a subgroup K_1 of ClK which is isomorphic to a circle group S^1 and $K_1(x)\not\subset G(x)$. Now we will deform the metric g so that the maps in K_1 are not isometries of the deformed metric. Let U be a neighborhood of x in M so small that $K_1(x)\backslash G(U)=\phi$, where $G(U)$ is the union of G-orbits of all the points in U. Let φ be a C^∞ function on M such that $\varphi\geq 0$, $\varphi(x)>0$ and $\mathrm{supp}(\varphi)\subset U$. If we define the function ψ by

$$\psi(y)=\int_{\alpha\in G}\varphi(\alpha y)dV_G,$$

where dV_G is a normalized biinvariant measure on G. Then $\psi(x)>0$, $\psi\geq 0$, $\mathrm{supp}(\psi)\subset G(U)$ and, for $t>0$, the tensors $g_t=(1+t\psi)\cdot g$ define Riemannian metrics on M. Moreover, as $t\to 0$, $g_t\to g$ in C^∞-topology.

Now, by (2.1), for sufficiently small $t>0$, $\mathrm{Isom}(g_t)$ is conjugate isomorphic to a subgroup F_t of $\mathrm{Isom}(g)$ by the isomorphism η_t : $\mathrm{Isom}(g_t)\to F_t$. The difference of the actions of $\mathrm{Isom}(g_t)$ and F can be described by

$$D_t=\sup\{\mathrm{dis}_{C^0}(\alpha,\eta_t(\alpha)) : \alpha\in\mathrm{Isom}(g_t)\}$$

and $D_t\to 0$ as $t\to 0$.

Since both ψ and g are invariant under the action of G, so is g_t and $G \subset \mathrm{Isom}(g_t)$. Moreover, if $\alpha \in K_1$ so that $y = \alpha(x) \notin U$, then, for $t > 0$,

$$g_t(x) = g(x) + t \cdot \psi(x) \cdot g(x) \neq g(x) = \alpha^*(g(y)) = \alpha^*(g_t(y))$$

which implies $\alpha \notin \mathrm{Isom}(g_t)$ for $t > 0$ and, therefore $\alpha \notin F_t$ for all sufficiently small $t > 0$, because $D_t \to 0$ as $t \to 0$. Thus we have constructed desired metrics g_t for t sufficiently small.

Remarks. According to this theorem, for the averages of each Riemannian metric over the actions of two compact Lie groups to have identical groups of isometries, it is necessary that the orbits of the two actions coincide for each point. This happens in an obvious way when a manifold is made from two distinct compact Lie groups as homogeneous spaces, in which case the only orbit is the whole space. For example, the standard actions of $SU(2)$ on S^3 induces the standard Riemannian metric on S^3. Therefore the standard action of $SU(2)$ on S^3 can never be the action of the full group of isometries, because once a Riemannian metric is invariant under $SU(2)$ so it is under the standard action of $SO(4)$.

Theorem 2. *Let G be compact Lie groups acting effectively on M and let H be a compact Lie subgroup of G so that the principal orbits of the actions of the two groups coincide for each point of M. If the averages g_G and g_H(resp.) of a Riemannian metric g over the actions of G and H(resp.) induce, on each principal orbit, Riemannian metrics with identical groups of isometries, then $g_G = g_H$ on M.*

Proof. The averaging process of the Riemannian metric over the action of G can be performed in two steps: i.e. first average over H and then over G/H where both spaces are equipped with normalized invariant measures. Now let $G(x) = H(x)$ be a principal orbit. A tangent vector X is perpendicular to the orbit in g_G, if for each Y tangent to the orbit

$$0 = \int_{\alpha \in G} g(\alpha_* X, \alpha_* Y) = \int_{\beta H \in G/H} \int_{\alpha \in H} g(\beta_* \alpha_* X, \beta_* \alpha_* Y)$$
$$= \int_{\beta H \in G/H} g_H(\beta_* X, \beta_* Y).$$

But this is true if

$$0=g_H(\alpha_* X, \alpha_* Y)$$

and hence the normal bundles for g_G and g_H to the orbit coincide. Moreover, the isotropy subgroup G_x leaves invariant each vector in the normal bundle N_x and hence every point in a normal slice. Since otherwise G_x will induce at least one dimensional motion in the slice (a rotation centered at x) which will result in an orbit of higher dimension than $G(x)$ through a point in the slice, contradicting the fact that $G(x)$ is principal.

Similar argument as in the beginning of the proof applies to the tangent vectors to the orbit and we have

$$g_G|G(x)=g_H|G(x).$$

Also both H_x and G_x leave N_x pointwisely fixed and averaging processes over G and H (resp.) are same as those over G/G_x and H/H_x(resp.). Hence we have

$$g_G|N_x=g_H|N_x$$

and, therefore $g_G=g_H$ at x.

Since principal orbits are dense in M, $g_G=g_H$ on M.

REMARK. Therefore to see whether a compact Lie transformation group is the full group of isometries of a Riemannian metric, one only needs to check whether there is a larger compact Lie transformation group with same orbit structures such that there is an orbit on which there is a Riemannian metric invariant under the original transformation group but not under the larger one. If there is such one it easily extends to an invariant metric on an invariant tube and then using the partitions of unity argument one can extend to the whole manifold.

Therefore one only needs to understand the behavior of transformation groups on orbits and thus the problem reduces to that of homogeneous spaces.

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