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HYPOELLIPTICITY OF SYSTEMS OF ANALYTIC VECTOR FIELDS

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1. Introduction

In this paper, we are concerned with the pointwise-hypoellipticity (see Definition 2. 1) of an m-dimensional Frobenious Lie algebra L of analytic complex vector fields in some open subset Ω of R^{m+1}. That is, L is a set of complex vector fields in Ω with (real-) analytic coefficients satisfying:

(A) each point of Ω has an open neighborhood in which L is generated by m linearly independent elements of L;

(B) L is closed under the commutation bracket $[A, B]$.

The pointwise-analytic hypoellipticity of L is completely characterized by M. S. Baouendi and F. Treves in [1]. Here, we shall prove that if L is hypoelliptic at a point then it must be analytic hypoelliptic in a full neighborhood of the same point. When the coefficients are C^∞, hypoellipticity of L was discussed in [2].

By shrinking Ω about a point in Ω, which we may take as the origin in R^{m+1}, we may choose a coordinate system $t_1, \cdots, t_m$, x in R^{m+1} so that L is spanned by m linearly independent vector fields $L_1, \cdots, L_m$ satisfying

(1. 1) $$L_j=\frac{\partial}{\partial t_j}+\lambda_j(t, x)\frac{\partial}{\partial x}, \quad j=1, \cdots, m,$$

where λ_j are (real-) analytic complex valued functions in Ω.

Let $Z(t, x)$ be the unique analytic solution of the Cauchy problem

(1. 2) $$L_jZ=0, \quad j=1, \cdots, \quad m \qquad Z|_{t=0}=x \text{ in } \Omega$$

which may take the form, after shrinking Ω,

(1. 3) $$Z=x+i\phi(t, x), \quad \phi \text{ real valued}, \quad \phi(0, x)=0.$$

We say that $Lu=0, u\in D'(\Omega)$ if $Mu=0$ for all vector field M in L, which is equivalent to saying that $L_ju=0$ in Ω for all $j=1, \cdots, m$.

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2. Pointwise-hypoellipticity

DEFINITION 2.1. The Lie algebra L is analytic hypoelliptic at a point w_0 of Ω if given any distribution u in an open neighborhood $U\subset\Omega$ of w_0 such that, given any vector field $M\in L$, Mu is analytic in some open neighborhood $V\subset U$ of w_0, then u is analytic in some open neighborhood $U_0\subset U$ of w_0.

We shall say that L is analytic hypoelliptic in a subset A of Ω if this is so at every point of A, which is the same as the usual analytic hypoellipticity, when A is open. We shall also use the C^∞ analogues of these definitions.

In [1], M.S. Baouendi and F. Treves characterized the pointwise-analytic hypoellipticity of L by using a local constancy principle of L. Let r be a positive number and J be an open interval containing the origin such that the closure $\overline{B_r\times J}$ of $B_r\times J$, $B_r=\{t\in R^m : |t|<r\}$, is contained in Ω.

PROPOSITION 2.1. (Baouendi and Treves [1]) *The following properties of L are equivalent.*

(*A*) *L is not analytic hypoelliptic at the origin.*

(*B*) *There is a number r', $0<r'<r$ such that zero is not an interior point of the image of $B_{r'}$ under the map $t\mapsto\phi(t,0)$.*

(*C*) *There is an open neighborhood $V\subset\Omega$ of the origin such that, given any integer $k\geq 0$, there is an $u\in C^k(V)$ satisfying $Lu=0$, but not of class C^{k+1} in any subneighborhood of the origin.*

In particular, if L is hypoelliptic at a point in Ω, it is analytic hypoelliptic at the same point. But there is an example of a single vector field in R^2 which is analytic hypoelliptic but not hypoelliptic at the origin [1].

THEOREM 2.1. *If L is hypoelliptic at a point in Ω, then it is analytic hypoelliptic in some neighborhood of the same point.*

Proof. It suffices to show the theorem at the origin. If we assume that any neighborhood of the origin contains a point at which L is not analytic hypoelliptic, then there is a sequence $\{w^i\}$, $w^i=(t^i, x^i)$, of points in Ω which converges to the origin and at which L is not analytic hypoelliptic. Since L is not analytic hypoelliptic at each w^i, by proposition 2.1, there is a positive number r^i, such that $\phi(w^i)$ is not an interior point of the image of B_{r^i}, open ball with center t^i and radius r^i, under the map $t \mapsto \phi(t, x^i)$. So if J^i is an open interval containing x^i such that $\overline{B_{r^i} \times J^i} \subset \Omega$, then $Z(B_{r^i} \times J^i)$ does not intersect at least one of the two open rays $Re\, Z = x^i$, $Im\, Z > \phi(w^i)$, and $Re\, Z = x^i$, $Im\, Z < \phi(w^i)$, so that one can define a single valued branch of $Z^{3/2}$ on $Z(B_{r^i} \times J^i)$. For each i, select a function $g_i \in C_0^\infty(B_{r^i} \times J^i)$ equal to one in a neighborhood of w^i and whose support does not contain any point w^j for $j \neq i$. The continuous function $u_i(t, x) = g_i(t, x)[Z(t, x) - Z(w^i)]^{3/2}$ satisfies $L_j u_i \in C_0^\infty(R^{m+1})$, for all $j = 1, \cdots, m$. We can find a sequence of numbers $c_i > 0$ such that $\sum c_i u_i$ converges uniformly to a compactly supported continuous function u and that $\sum c_i L_j u_i$ converges in $C_0^\infty(R^{m+1})$, for all $j = 1, \cdots, m$. But the singular support of u must contain the origin since u is singular at each w^i, whence L can not be hypoelliptic at the origin. This completes the proof.

If m is equal to one, that is, if L is generated by a single complex vector field, then due to the result in [3] the hypoellipticity and analytic hypoellipticity for L are equivalent. So we have the following corollary:

Corollary 2.1. *For any complex analytic vector field L in R^2, the followings are equivalent.*

(A) L is hypoelliptic at the origin.

(B) There is a neighborhood of the origin in which L is analytic hypoelliptic.

(C) There is a neighborhood of the origin in which L is hypoelliptic.

References

1. M.S. Baouendi and F. Treves, *A local constancy principle for the solution of certain overdetermined systems of first-order linear P.D.E.*, Advances in Math. Supplementary Studies, Vol. 7 A in honor of L. Schwartz (1981), 245-262.

2. K.H. Kwon and B.C. Song, *Hypoellipticity of Locally Integrable Systems of Complex Vector Fields*, Preprint.
3. F. Treves, *Analytic Hypoelliptic Partial Differential Equations of Principal Type*, Comm. Pure Appl. Math. **24**(1971), 537-570.

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