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SOME PERMANENTAL INEQUALITIES

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1. Introduction

Let Ω_n and $\mathrm{Pm}t_n$ denote the sets of all $n\times n$ doubly stochastic matrices and the set of all $n\times n$ permutation matrices respectively. For $m\times n$ matrices $A=[a_{ij}]$, $B=[b_{ij}]$ we write $A\leq B(A<B)$ to mean that $a_{ij}\leq b_{ij}(a_{ij}<b_{ij})$ for all $i=1,\cdots,m$; $j=1,\cdots,n$. Let I_n denote the identity matrix of order n, let J_n denote the $n\times n$ matrix all of whose entries are $1/n$, and let $K_n=nJ_n$. For a complex square matrix A, the permanent of A is denoted by per A. Let E_{ij} denote the matrix of suitable size all of whose entries are zeros except for the (i,j)-entry which is one.

For an $n\times n$ matrix A and for $i_1,\cdots i_s$, $j_1,\cdots,j_s\in\{1,\cdots,n\}$, let $A(i_1,\cdots,i_s|j_1,\cdots,j_s)$ denote the matrix obtained from A by deleting the rows $i_1,\cdots,i_s$ and the columns $j_1,\cdots,j_s$.

For positive integral n-vectors R, S and nonnegative $n\times n$ matrices A, B, let $U_{R,S}(A,B)$ denote the set of all $n\times n$ matrices X whose row sum vector and column sum vector are R and S respectively and such that $A\leq X\leq B$.

The sets $U_{R,S}(A,B)$with $A\leq B\leq K_n$ have been studied in [1] as faces of the so called assignment polytope $U_{R,S}(O,K_n)$.

In general, it is very hard to determine the minium and the maximum values of permanent function on the set $U_{R,S}(A,B)$ even with some good restrictions on the vectors R and S as well as on the bound matrice A and B.

A particular case of $U_{R,S}(A,B)$ with $R=S=(1,\cdots,1)$, $A=O$, $B=K_n$ is the set Ω_n, on which the minimum permanent is achieved uniquely at J_n. This result was conjectured in 1926 by van der Waerden and proved by Egorycev in 1980, and now is called the van der Waerden-Egorycev's theorem in the literature.

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Marcus and Minc [6] conjectured that, for any $A \in \Omega_n$, $n \geq 2$,

$$\operatorname{per}\left(\frac{nJ_n - A}{n-1}\right) \leq \operatorname{per} A$$

with equality if and only if $A = J_n$ for $n \geq 4$, and proved their conjecture for positive semi-definite symmetric doubly stochastic matrices.

In [7], E.T.H. Wang proved Marcus-Minc' conjecture for the case of $n=3$ and proposed a conjecture asserting that

$$\operatorname{per}\left(\frac{nJ_n + A}{n+1}\right) \leq \operatorname{per} A$$

for all $A \in \Omega_n$, $n \geq 2$.

Marcus-Minc' conjecture was proved to be true for the case of $n=4$ by T. Foregger [4]. Both of these two conjectures are true for doubly stochastic matrices in a sufficiently small neighbourhood of J_n because of the van der Waerden-Egorycev's theorem. Recently, D.K. Chang [2], [3] has proven the validity of these two conjectures in the complement of a sufficiently large neighbourhood of J_n by showing first that

(1) $$\operatorname{per}\left(\frac{nJ_n - A}{n-1}\right) \leq \frac{d_n}{(n-1)^n}$$

where d_n denotes the n-th derangement number $n! \sum_{k=0}^{n} \frac{(-1)^k}{k!}$, and

(2) $$\operatorname{per}\left(\frac{nJ_n + A}{n+1}\right) \leq \operatorname{per}\left(\frac{nJ_n + I_n}{n+1}\right)$$

for all $A \in \Omega_n$, $n \geq 2$.

Let $E_n = (1, \cdots, 1)$, the n-tuple of ones. In this paper we determine the set of permanent-maximal matrices in $(U_{R,S}(A, B)$ along with the maximum value of the permanent function for the case of $R=S=(n-1)E_n$, $A=O$, $B=K_n$ or of $R=S=(n+1)E_n$, $A=K_n$, $B=2K_n$, by a simple combinatorial argument. And as a corollary, we will have the following permanental inequality;

(3) $$\operatorname{per}\left(\frac{nJ_n \pm A}{n \pm 1}\right) \leq \frac{n!}{(n \pm 1)^n} \sum_{k=0}^{n} \frac{(\pm 1)^k}{k!}$$

for all $A \in \Omega_n$, $n \geq 2$, where the signs $+$, $-$ occur in the same order.

Notice that the inequality (3) is the conbination of the inequalities (1) and (2). In addition, we also show that equality in (3) holds if and only if $A \in \mathrm{Pmt}_n$, for $n \geq 4$.

Finally, we find out some subclass of Ω_n over which the two conjecture are valid.

2. Maximum permanent on $U_{R,S}(A, B)$

In this section, we obtain the maximum value of permanent on $U_{R,S}(A, B)$ and find out the matrices in $(U_{R,S}(A, B)$ at which the maximum value is achieved for the case of $R=S=(n-1)E_n$, $A=O$, $B=K_n$ or of $R=S=(n+1)E_n$, $A=K_n$, $B=2K_n$.

For this purpose only, let

$$U_{n-1}=\{X=[x_{ij}] \mid \sum_{j=1}^{n} x_{ij}=n-1 \ \ (i=1, \cdots, n), \ \ O \leq X \leq K_n\},$$

$$U_{n+1}=\{X=[x_{ij}] \mid \sum_{j=1}^{n} x_{ij}=n+1 \ \ (i=1 \cdots, n), \ \ K_n \leq X \leq 2K_n\}.$$

Lemma 1. *The permanent function attains its maximum on each of U_{n-1} and U_{n+1} at an integral matrix.*

Proof. Let $A \in U_{n-1}$ be such that per $A \geqq$ per X for all $X \in U_{n-1}$ with as few non integral entries as possible.

Suppose that A is not an integral matrix. Then there are i, j such that $0<a_{ij}<1$. Since the i-th row sum of A is an integer, there is some l, $l \neq j$, such that $0<a_{il}<1$. For a real number ε with sufficiently small absolute value, let $A_\varepsilon = A+\varepsilon(E_{ij}-E_{il})$. Then $A_\varepsilon \in U_{n-1}$ and

$$\operatorname{per} A_\varepsilon = \operatorname{per} A + \varepsilon(\operatorname{per} A(i|j) - \operatorname{per} A(i|l)).$$

Hence it follows that per $A(i|j)=$ per $A(i|l)$. Therefore $f(\varepsilon)=$ per A_ε is a constant function of ε. Now, by choosing a suitable ε, we come to get a permanent-maximal matrix $A_\varepsilon \in U_{n-1}$ with strictly fewer non-integral entries than A, contradicting the choice of A. Thus the assertion for U_{n-1} is proved.

Similarly, we can prove the lemma for U_{n+1}.

Lemma 2. *Let U^*_{n-1}, U^*_{n+1} denote the sets of all integral matrices in U_{n-1}, U_{n+1} respectively. Then*

(1) *For all* $A \in U_{n-1}^*$, $\operatorname{per} A \le n! \sum_{k=0}^{n} \frac{(-1)^k}{k!}$, *with equality if and only if* $A = K_n - P$ *for some* $P \in \mathrm{Pmt}_n$, *for* $n \ge 4$.

(2) *For all* $A \in U_{n+1}^*$, $\operatorname{per} A \le n! \sum_{k=0}^{n} \frac{1}{k!}$, *with equality if and only if* $A = K_n + P$ *for some* $P \in \mathrm{Pmt}_n$.

Proof. (1) Let $U = [u_{ij}] \in U_{n-1}^*$ be such that $\operatorname{per} X \le \operatorname{per} U$ for all $X \in U_{n-1}^*$. Notice that every row of U is an n-vector all of whose components are 1's except for exactly one which is 0. Let $C = (c_1, \cdots, c_n)$ be the column sum vector of U. Without loss of generality, we may assume that $c_1 = \max\{c_1, \cdots, c_n\}$ and $c_2 = \min\{c_1, \cdots, c_n\}$. Then $c_1 \ge n-1$ and $c_2 \le n-1$.

Suppose that $c_1 > n-1$. Then $c_1 = n$ so that $u_{11} = \cdots = u_{n1} = 1$. Since, in this case, $c_2 \le n-2$, we may also assume that $u_{12} = u_{22} = 0$. Then it follows that $\operatorname{per} U(1|1) \le \operatorname{per} U(1|2)$. Moreover if $n \ge 4$, $\operatorname{per} U(1|1) < \operatorname{per} U(1|2)$. For, if $n \ge 5$, then $\operatorname{per} U(1,2|1,2) > 0$ by Frobenius-König's theorem. If $n = 4$, $\operatorname{per} U(1,2|1,2) = 0$ only if $U(1,2|1,2)$ has a zero column so that U must be permutation equivalent to

$$\begin{bmatrix} 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

But, if it is the case, $\operatorname{per} U = 8 < 9 = \operatorname{per}(K_3 - I_3)$, which is impossible because of the choice of U and because $K_3 - I_3 \in U_{n-1}$.

Let $H = U - E_{11} + E_{12}$. Then $H \in U_{n-1}^*$ and $\operatorname{per} H \ge \operatorname{per} U$. If $n \ge 4$, we have $\operatorname{per} H > \operatorname{per} U$, contradicting the maximality of $\operatorname{per} U$. Therefore it must be that $c_1 = n-1$ so that $c = (n-1)E_n$, in which case it must be that $U = K_n - P$ for some $P \in \mathrm{Pmt}_n$.

In the case $n = 3$, if the column sum vector of H is different from $2E_3$, then we do the same job as above to get a matrix $M \in U_{n-1}^*$ with row sum vector $2E_3$ and $\operatorname{per} M \ge \operatorname{per} H$. But then $M = K_3 - P$ for some $P \in \mathrm{Pmt}_3$ and $\operatorname{per} M = \operatorname{per} U$ by the maximality of $\operatorname{per} U$. Now since $\operatorname{per}(K_n - P) = \operatorname{per}(K_n - I_n) = n! \sum_{k=0}^{n} [(-1)^k / k!]$ for all $P \in \mathrm{Pmt}_n$, the proof is completed.

(2) Again let $U\in U_{n+1}^{*}$ be such that per $X\leq$ per U for all $X\in U_{n+1}^{*}$ and let $C=(c_1,\cdots,c_n)$ be the column sum vector of U. Notice, in this case, that every row of U is an n-vector, of the components of which $n-1$ are 1's and one is 2. Since U is a positive matrix, every submatrix of U has positive permanent. With these facts in mind, we can show that $C=(n+1)E_n$ regardless of whether $n>3$ or not by a similar argument as the one used in the proof of (1). But then $U=K_n+P$ for some $P\in\mathrm{Pmt}_n$. Now the assertion (2) follows because $\mathrm{per}(K_n+P)=\mathrm{per}(K_n+I_n)=n!\sum_{k=0}^{n}(1/k!)$ for all $P\in\mathrm{Pmt}_n$.

Since, for every $P\in\mathrm{Pmt}_n$, $K_n-P\in U_{(n-1)E_n,(n+1)E_n}(O,K_n)$ and $K_n+P\in U_{(n+1)E_n,(n+1)E_n}(K_n,2K_n)$, the combination of lemmas 1 and 2 tells us that, for every $A\in U_{(n-1)E_n,(n-1)E_n}(O,K_n)$, per $A\leq\mathrm{per}(K_n-P)$, and for every $A\in U_{(n+1)E_n,(n+1)E_n}(K_n,2K_n)$, per $A\leq\mathrm{per}(K_n+P)$. Thus we have proven half of the following

THEOREM 3. (1) *For any* $A\in U_{(n-1)E_n,(n-1)E_n}(0,K_n)$,

$$\mathrm{per}\,A\leqq n!\sum_{k=1}^{n}\frac{(-1)^k}{k!}$$

with equality if and only if $A=K_n-P$ *for some* $P\in\mathrm{Pmt}_n$, *if* $n\geqq4$.

(2) *For any* $A\in U_{(n+1)E_n,(n+1)E_n}(K_n,2K_n)$,

$$\mathrm{per}\,A\leq n!\sum_{k=0}^{n}\frac{1}{k!}$$

with equality if and only if $A=K_n+P$ *for some* $P\in\mathrm{Pmt}_n$.

Proof. We need only to show that, for $n\geqq4$, every permanent-maximal matrix on U_{n-1} is a $(0,1)$-matrix. Suppose that there is a non $(0,1)$ permanent-maximal matrix $A=[a_{ij}]$ in U_{n-1}, then we can pick up such an A with exactly two non integral entries which are in a same row, say a_{11} and a_{12}. Then, by Lemma 2, we may assume that

$$A=\left[\begin{array}{c|cccc} a_{11} & a_{12} & 1 & \cdots & 1 \\ \hline 1 & & & & \\ \vdots & & K_{n-1}-I_{n-1} & & \\ 1 & & & & \end{array}\right]$$

showing us that $\operatorname{per} A(1|1) < \operatorname{per} A(1|2)$ and hence that for ε, $0 < \varepsilon < \min\{a_{11}, a_{12}\}$, $\operatorname{per}(A+\varepsilon(E_{12}-E_{11})) > \operatorname{per} A$ even if $A+\varepsilon(E_{12}-E_{11}) \in U_{n-1}$, a contradiction. Therefore there can not be such an A and the proof is completed.

Since $\left\{\frac{1}{n-1}A \mid A \in U_{(n-1)E_n,(n-1)E_n}(O, K_n)\right\} = \left\{\frac{K_n - S}{n-1} \mid S \in \Omega_n\right\}$ and since $\left\{\frac{1}{n+1}A \mid A \in U_{(n+1)E_n,(n+1)E_n}(K_n, 2K_n)\right\} = \left\{\frac{K_n + S}{n+1} \mid S \in \Omega_n\right\}$, We have the following

COROLLARY. *For any* $S \in \Omega_n$,

(1) $$\operatorname{per}\left(\frac{nJ_n - S}{n-1}\right) \leq \frac{n!}{(n-1)^n} \sum_{k=0}^{n} \frac{(-1)^k}{k!}$$

with equality if and only if $S \in \mathrm{Pmt}_n$ *if* $n \geq 4$, *and*

(2) $$\operatorname{per}\left(\frac{nJ_n + S}{n+1}\right) \leq \frac{n!}{(n+1)^n} \sum_{k=0}^{n} \frac{1}{k!}$$

with equality if and only if $S \in \mathrm{Pmt}_n$.

3. Permanents of partly decomposable matrices

In this section, we are to show that every partly decomposable doubly stochastic matrix satisfies the Marcus-Minc conjecture and Wang's conjecture.

An $n \times n$ matrix is called *partly decomposable* if it contains an $s \times t$ zero submatrix with $s+t=n$.

Let D be an $n \times n$ $(0,1)$-matrix with $\operatorname{per} D \neq 0$, then $\Omega(D) = \{X \in \Omega_n \mid X \leq D\}$ is a face of Ω_n [5].

Let p, q be positive integers such that $p+q=n$. Suppose that $D = \begin{bmatrix} K_p & 0 \\ * & K_q \end{bmatrix}$ is a $(0,1)$-matrix. It is known that $J_p \oplus J_q$ is the unique permanent-minimal matrix on $\Omega(D)$. (see [5], for example).

From now on, in the sequel, let $\delta^k = k!/k^k$ for $k = 1, 2, \cdots$.

THEOREM 4. *Let A be any partly decomposable matrix in Ω_n, then* $\operatorname{per} A \geq \delta_{n-1}$ *with equality if and only if* $A = P(I_1 \oplus J_{n-1})Q$ *for some* $P, Q \in \mathrm{Pmt}_n$.

Proof. Let $A\in\Omega_n$ be partly decomposable such that per $A\leq$per X for all partly decomposable $X\in\Omega_n$. Then $A=J_p\oplus J_q$ for some integers p,q with $p,q\geq 1$ and $p+q=n$, so that per $A=\delta_p\delta_q$. We are to show that either $p=1$ or $q=1$. Suppose that $q\geq p>1$. Then

$$\delta_{p-1}\delta_{q+1}=\left(\frac{p}{p-1}\right)^{p-1}\left(\frac{q}{q+1}\right)^{q}\delta_p\delta_q<\delta_p\delta_q$$

showing us that $\mathrm{per}(J_{p-1}\oplus J_{p+1})<\mathrm{per}(J_p\oplus J_q)$. Therefore it must be that $p=1$ by the minimality of $J_p\oplus J_q$, and we are done.

To prove the validity of Marcus-Minc conjecture and Wang's conjecture for partly decomposable doubly stochastic matrices, it suffices to show only that

$$\frac{n!}{(n\pm 1)^n}\sum_{k=0}^{n}\frac{(\pm 1)^k}{k!}\leq\delta_{n-1}$$

where the signs $+$, $-$ are written in the same order. But since

$$\sum_{k=0}^{n}\frac{(-1)^k}{k!}\leq\sum_{k=0}^{4}\frac{(-1)^k}{k!}=\frac{3}{8}\quad\text{and}\quad\frac{3}{8}\,\frac{n}{n-1}<1\ \text{for}\ n\geq 4,$$

we have

$$\frac{n!}{(n-1)^n}\sum_{k=0}^{n}\frac{(-1)^k}{k!}\leq\frac{3}{8}\,\frac{n}{n-1}\delta_{n-1}<\delta_{n-1}.$$

On the other hand, we have, for all $n>2$,

$$\left(\frac{n+1}{n}\right)^n\left(\frac{n}{n-1}\right)^{n-1}\geq\left(\frac{3}{2}\right)^2\left(\frac{2}{1}\right)^1=4.5>e$$

that is

$$\left(\frac{n}{n+1}\right)^n e<\left(\frac{n}{n-1}\right)^{n-1}.$$

Thus we have

$$\frac{n!}{(n+1)^n}\sum_{k=0}^{n}\frac{1}{k!}<\left(\frac{n}{n+1}\right)^n e\delta_n<\delta_{n-1}$$

since $\delta_{n-1}=\left(\dfrac{n}{n-1}\right)^{n-1}\delta_n$.

Since Marcus-Minc conjecture is already known to be true for $n=3$, by the above discussions we have the following

THEOREM 5. *For any partly decomposable* $S \in \Omega_n$, $n \geq 2$,

$$\operatorname{per}\left(\frac{nJ_n \pm S}{n \pm 1}\right) \leq \operatorname{per} S$$

where the signs $+, -$ *are written in the same order.*

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