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BLOCH SPACES IN THE BALL AND HANKEL OPERATORS

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1. Preliminaries

Let B be the unit ball in C^n and $H(B)$ be the space of holomorphic functions in the ball, for $z=(z_1, \cdots, z_n)$ and $\alpha=(\alpha_1, \cdots, \alpha_n) \in Z_+^n$, we write $z^\alpha = z_1^{\alpha_1} \cdots z_n^{\alpha_n}$ and $\alpha! = \alpha_1! \cdots \alpha_n!$ and $|\alpha| = \alpha_1 + \cdots + \alpha_n$, For $z, \zeta \in C^n$, $\langle z, \zeta \rangle$ denotes $z_1\zeta_1 + \cdots + z_n\zeta_n$ while $z \cdot \zeta$ means $(z_1\zeta_1, \cdots, z_n\zeta_n)$. Let dv be the Lebesgue measure on C^n and let $d\sigma$ be the normalized surface measure on the boundary ∂B of B. For $q>0$, dv_q is the probability measure on $\bar{B}$ defined by

$$dv_q(z) = \frac{\Gamma(n+q)}{\pi^n \Gamma(q)} (1-\|z\|^2)^{q-1} dv(z),$$

while dv_0 is, by definition, identified with $d\sigma$. For $r, n \in \boldsymbol{R}^+$, $(r)_n = r(r+1) \cdots (r+n+1) = \Gamma(n+r)/\Gamma(r)$. With these notations, we recall the weighted Bergman spaces in the ball[2].

$$A_q^p = \{f \in H(B) : \|f\|_{p,q}^p = \int |f|^p dv_q < \infty\}.$$

Since we only need the case for which $q=0$, $q=1$ in this paper, We shall use rather simplified notation: $A^2 \equiv A_1^2$, $\|f\|_p \equiv \|f\|_{p,1}$. We also let $\langle f, g \rangle_1 \equiv \int f g dv_1$ whenever it makes sense. Finally, we remind the following formula:

$$\int f dv_1 = 2n \int_0^1 r^{2n-1} \int_{\partial B} f(rz) d\sigma(z) dr, \quad f \in A_1^1.$$

We define

$$\mathscr{B} = \{f \in H(B),\ \|f\|_{\mathscr{B}} = \sup (1-\|z\|^2)\ |(R+n+1) f(z)| < \infty\},$$

where $R = \sum z_j \partial / \partial z_j$ is the radial derivative operator introduced by W. Rudin[5].

We also define $\mathscr{B}_0$ to be the set of all $f \in \mathscr{B}$ with $\lim_{z \to 1^-} (1-\|z\|^2) | (R+n+1) f(z) | = 0$. For reference, we list other norms introduced

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by Hahn[4], Timoney[6].

$$\|f\|_H=\sup Q_f(z),\ \text{where } Q_f(z)=\sup_{\|\zeta\|=1}\frac{|\langle\nabla f,\zeta\rangle|}{\sqrt{1-\|z\|^2+|\langle z,\zeta\rangle|^2}}$$

$$\|f\|_R=\sup(1-\|z\|^2)|Rf(z)|$$

$$\|f\|_G=\sup(1-\|z\|^2)|\nabla f(z)|$$

In [6], it is shown that above norms are equivalent (under quotient) and we note that those are again equivalent to $\|\cdot\|_{\mathscr{B}}$, since $\|f\|_R<\infty$ or $\|f\|_{\mathscr{B}}<\infty$ implies $\sup(1-\|z\|^2)|f(z)|<\infty$. One difference is that $\|\cdot\|_{\mathscr{B}}$ is an actual norm while the others are only seminorms.

2. Duality in Bloch spaces

THEOREM 1.1.

(1.1) *$\mathscr{B}$ is a Banach space.*

(1.2) *$\mathscr{B}_0$ is the $\mathscr{B}$-closure of polynomials, hence separable.*

(1.3) *f is in $\mathscr{B}$ if and only if*
$R=\sup\{$radii of schlicht disks in the range of $f\}<\infty$.

(1.4) *f is in $\mathscr{B}$ if and only if for each $\zeta\in\partial B$, f_ζ is a Bloch function in one variable sense.*

(1.5) *For $n=1$, $\mathscr{B}_0{}^*=D$ and $D^*=\mathscr{B}$, where D is the Dirichlet space $\{f$ analytic in B_1 and $\iint|f(re^{i\theta})|rdrd\theta<\infty\}$.*

For the proof, we refer to [1,6].

Let A be the Dirichlet space in the ball defined by

$$A=\{f\in H(B),\ \|f\|_A=\int_B|(R+n)f|dv_1<\infty\}.$$

We note that this kind of Sobolev spaces were extensively studied by Burbea[2]. The next theorem, which is an n-dimensional version of (1.5) also appeared in [6], but was not proved explicitly. In chapter 1, we give the proof for the sake of completeness and obtain new estimates with our norm, and in chapter 2, we study the compactness of Hankel operators.

THEOREM 1.2. *$\mathscr{B}$ is isomorphic to the dual of A and A is isomorphic to the dual of $\mathscr{B}_0$. More specifically, each $f(z)=\sum a_\alpha z^\alpha\in\mathscr{B}$ is a continuous linear functional on A via*

(1.6) $$T_f(g)=\lim_{\rho\to 1}\sum \frac{n\alpha!}{n_{|\alpha|}} a_\alpha b_\alpha \rho^{|\alpha|}=\lim_{\rho\to 1^-}\langle g_\rho, f\rangle_1,$$

where $g(z)=b_\alpha z^\alpha \in A$. Conversely, for any φ in A^* (ψ in $\mathscr{B}_0{}^*$), there exists a unique f in $\mathscr{B}$ (g in A) such that $T_f=\varphi$ ($S_g=\psi$). Furthermore, there exists a constant C such that

(1.7) $$C\|f\|_{\mathscr{B}}\leq \|T_f\|_{A^*}\leq \|f\|_{\mathscr{B}},$$

and

(1.8) $$C\|g\|_A\leq \|S_g\|_{\mathscr{B}^*}\leq \|g\|_A,$$

where $S_g(f)=\lim_{\rho\to 1^-}\langle f_\rho, g\rangle, f\in \mathscr{B}_0$.

Proof. For $|\zeta|<1$, we consider

$$\int (1-\|z\|^2)(R+n)g(z)(R+n+1)f(\zeta\cdot z)dv$$
$$=2n\int r^{2n}(1-r^2)(\sum (|\alpha|+n)b_\alpha z^\alpha r^{|\alpha|})$$
$$\times(\sum (|\alpha|+n+1)a_\alpha z^\alpha \zeta^\alpha r^{|\alpha|})d\sigma(z)dr.$$
$$=2n\sum\int (r^{2|\alpha|+2n-1}-r^{2|\alpha|+2n+1})(|\alpha|+n+1)(|\alpha|+n)\frac{\alpha!}{n_{|\alpha|}}a_\alpha b_\alpha \zeta^\alpha dr$$
$$=\sum \frac{n\alpha!}{n_{|\alpha|}}a_\alpha b_\alpha \zeta^\alpha.$$

Hence $\left|\sum \frac{n\alpha!}{n_{|\alpha|}}a_\alpha b^\alpha \zeta^\alpha\right|\leq \sup(1-\|z\|^2)\,|(R+n+1)f(\zeta\cdot z)|\cdot\|g\|_A$ $=\|f\|_{\mathscr{B}}\cdot\|g\|_A$. Now we let $\rho\to 1^-$ so that $\zeta=(\rho,\cdots,\rho)\to(1,\cdots,1)$ and we have

$$|T_f(g)|\leq \|f\|_{\mathscr{B}}\cdot\|g\|_A,$$

which proves (1.6).

Conversely, suppose φ is a continuous linear functional on A. We let $\varphi(z^\alpha)=\frac{n\alpha!}{n_{|\alpha|}}a_\alpha$ and $f(\zeta)=\sum a_\alpha \zeta^\alpha$. Then, for $g(z)=\sum b_\alpha z^\alpha \in A$,

$$\varphi(g)=\sum b_\alpha \varphi(z^\alpha)=\lim_{\rho\to 1^-}\sum \frac{n\alpha!}{n_{|\alpha|}}a_\alpha b_\alpha \rho^{|\alpha|}=T_f(g).$$

To show that f belongs to $\mathscr{B}$, consider

$$\sum (|\alpha|+n+1)a_\alpha z^\alpha=(R+n+1)f(z)$$
$$=\lim_{\rho\to 1^-}\left\{\sum \frac{\Gamma(n+|\alpha|+2)}{\alpha!\Gamma(n+1)}z^\alpha \zeta^\alpha \rho^{|\alpha|}, f(\zeta)\right\}$$
$$=\varphi\left(\sum \frac{\Gamma(n+|\alpha|+2)}{\alpha!\Gamma(n+1)(n+|\alpha|)}z^\alpha \zeta^\alpha\right)$$

$$=\varphi(F(\langle z,\zeta\rangle)),$$

where $F(t)=\sum\frac{(n+m+1)!}{n!m!(n+m)}t^m$.

To estimate the norm of $F(\langle z,\cdot\rangle)$, we note that

$$\|F(\langle z,\cdot\rangle)\|_A=\|\sum\frac{(n+m+1)!}{n!m!}\langle z,\cdot\rangle^m\|_1$$

$$=(n+1)\||1-\langle z,\cdot\rangle|^{-n-2}\|_1\sim(1-\|z\|)^{-1}.$$

Hence $(1-\|z\|^2)|(R+n+1)f(z)|\le C\|\varphi\|_{A^*}$ shows that $f\in\mathscr{B}$ and (1.7). The assertions about $\mathscr{B}_0{}^*$ can be proved similarly using (1.2) and an application of Hahn-Banach Theorem.

Remark. To find a good norm estimate one needs to compute the value

$$1/C=\sup_{z\in B}(n+1)(1-\|z\|^2)\||1-\langle z,\cdot\rangle|^{-n-2}\|_1.$$

This number will appear again in describing the Hankel operators.

2. Hankel operators in the ball

Definition. A Hankel operator T in A_1^2 with coefficients $\{a_\alpha\}$ is a linear operator with domain and range in A_1^2 such that

$$\langle Tz^\alpha,z^\beta\rangle=a_{\alpha+\beta}.$$

For $g(z)=\sum b_\alpha z^\alpha\in L^2(\partial B)$, the operator K_g defined by

$$K_g(f)=P(gf),\quad f\in A_1^2$$

is a Hankel operator with coefficients $\{b_\alpha\}$, where P is the orthgonal projection of $L^2(\partial B)$ onto the Hardy space H^2. To see this, we observe, for h,k in A_1^2,

$$\langle K_g(h),k\rangle_1=\langle P(gh),k\rangle_1=\langle gh,k\rangle_1=\langle ghk,1\rangle_1.$$

Hence, $\langle K_g(z^\alpha),z^\beta\rangle_1=\langle gz^{\alpha+\beta},1\rangle_1=b_{\alpha+\beta}$. A partial converse of the above fact is also true, more specifically, if φ is a Hankel operator in A_1^2 with coefficients $\{a_\alpha\}$ then

$$g(z)=\sum\frac{(n+1)_{|\alpha|}}{\alpha!}a_\alpha z^\alpha$$

belongs to A_1^2 and K_g has the same coefficients $\{a_\alpha\}$, because $g(z)=\varphi(z^0)$.

Theorem 2.1. *For $g\in A_1^2$, T_g becomes a bounded operator on A_1^2 if and only if g is in $\mathscr{B}$. Moreover, for the same constant C in* (1.7),

$$C\|g\|_{\mathscr{B}}\leq\|T_g\|\leq\|g\|_{\mathscr{B}}.$$

Proof. Let $g\in\mathscr{B}$, $u,v\in A_1^2$. Then $\langle T_g(u),v\rangle_1=\langle g,uv\rangle_1$. When f is in A^∞, we see, by the same computation as above, that

$$\langle f(z),(1-\|z\|^2)(R+n+1)g(z)\rangle_1=\langle f,g\rangle_1.$$

Hence for polynomials u,v we have

$$|\langle T_g(u),v\rangle_1|\leq\|g\|_{\mathscr{B}}\cdot\|uv\|_1\leq\|g\|_{\mathscr{B}}\|u\|_2\|v\|_2,$$

thus T_g extends to a bounded linear operator and $\|T_g\|\leq\|g\|_{\mathscr{B}}$. Conversely, let $\varphi_w(z)=(n+1)(1-\|w\|^2)(1-\langle z,w\rangle)^{-n-2}$ and $\psi_w(z)=\sqrt{\varphi_w(z)}$. We observe that

$$(2.1)\quad (1-\|w\|^2)(R+n+1)g(w)=\int_B g(z)\varphi_w(z)dv(z)=\langle g\varphi_w,1\rangle_1 =\langle T_g\psi_w,\psi_w\rangle_1.$$

Thus, it follows that

$$\|g\|_{\mathscr{B}}\leq\|T_g\|\cdot\|\varphi_w\|_1\leq 1/C\|T_g\|_{\mathscr{B}}.$$

Corollary. *Let $g\in A_1^2$ and $\{a_n\}$ be a dense sequence of points in B. Then T_g extends to a bounded operator on A_1^2 if and only if*

$$(2.2)\qquad \sup|\langle T_g\psi_{a_n},\psi_{a_n}\rangle_1|<\infty.$$

Proof. (2.2) implies, by (2.1) that g is a Bloch function, hence T_g is bounded. Conversely, if T_g is bounded, then (2.2) holds since $\|\psi_{a_n}\|_2\leq 1/C$.

Lemma. *If $f\in A_1^2$, then $\langle f,\psi_w\rangle_1\to 0$ as $\|w\|\to 1$.*

Proof. We write $f(z)=\sum a_\alpha z^\alpha$ *and* see that

$$|\langle f,\psi_w\rangle_1|$$
$$\leq C'\sqrt{(n+1)(1-\|w\|^2)}\cdot\sum_{m=0}^{\infty}\Big(\sum_{|\alpha|=m}\frac{\alpha!}{(n+1)_{|\alpha|}}|a_\alpha|^2\Big)^{1/2}\cdot\big(\sum_{|\alpha|=m}m!/\alpha!|w^{2\alpha}|\big)^{1/2}$$
$$\leq C''(1-\|w\|^2)^{1/2}\|f\|_2.$$

Hence $\langle f,\psi_w\rangle_1\to 0$ as $\|w\|\to 1$.

Theorem 2.2. *Let $g\in A_1^2$. Then T_g extends to a compact operator on A_1^2 if and only if $g\in\mathscr{B}_0$.*

Proof. For any polynomial p, T_p is compact. But by (1.2), there exist polynomials p_n such that $\lim\|g-p_n\|_{\mathscr{B}}=0$. Hence by (1.8), we have $\|T_g-T_{p_n}\|\leq\|g-p_n\|_{\mathscr{B}}$. Since compact operators form a closed subspace, it follows that T_g is compact. Conversely, let

T_g be compact. Then there exist orthonormal bases $\{u_n\}$, $\{v_n\}$ and eigenvalues $\lambda_n \to 0$, $\lambda_n \le \|T_g\|$ *and*

$$T_g f = \sum \lambda_n \langle f, v_n \rangle_1 u_n, \quad f \in A_1^2.$$

Then by (2.1),

$$(1-\|w\|^2)\,|(R+n+1)g(w)| \le |\sum \lambda_n \langle \psi_w, v_n \rangle_1 \langle u_n, \psi_w \rangle_1|$$
$$\le \|T_g\| \sum_1^N \langle \psi_w, v_n \rangle_1 \langle u_n, \psi_w \rangle_1| + \varepsilon \sum_{N+1}^{\infty} |\langle \psi_w, v_n \rangle_1 \langle u_n, \psi_w \rangle_1|.$$

Now the result follows from the lemma.

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