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A Pair of Commuting Operaters on Hilbert Spaces

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1. Introduction

Since the concept of joint spectrum for a family of operators was initially introduced by R. Arens and A.P. Calderon [1], several authors have established its definitions and properties. The typical and successful definitions among them have carried out by J.L. Taylor [8] and A.T. Dash [6].

In this paper we give a characterization of the joint spectrum, in the sense of J.L. Taylor of a pair of commuting operators on Hilbert spaces and some applications are given.

Let H be a complex Hilbert space and $B(H)$ the algebra of all linear continuous operators on H. Let $a=(a_1, a_2)\subset B(H)$ be a pair of commuting operators. Consider the sequence

$$(1.1)\qquad 0 \longrightarrow H \xrightarrow{\delta_a^0} H\oplus H \xrightarrow{\delta_a^1} H \longrightarrow 0,$$

where $\delta_a^0(x)=a_1x\oplus a_2x \ (x\in H)$ and $\delta_a^1(x_1\oplus x_2)=a_1x_2-a_2x_1(x_1,x_2\in H)$. Clearly, $a_1a_2=a_2a_1$ implies $\delta_a^1\cdot\delta_a^0=0$. Then, J.L. Taylor has defined a to be nonsingular if the sequence (1.1) is exact; i.e. im $\delta_a^0=$ ker δ_a^1. And he has defined the joint spectrum $\sigma(a,H)$ of a on H, to be the complement of the set of all $z-a=(z_1-a_1, z_2-a_2)$ is nonsingular on H.

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2. Invertibility of a commuting pair

We begin the following. Suppose that $a=(a_1,a_2)\subset B(H)$ is nonsingular on H. Consider the dual sequence of (1.1), namely

$$(2.1)\qquad 0\longrightarrow H\xrightarrow{\delta_a^{1*}}H\oplus H\xrightarrow{\delta_a^{0*}}H\longrightarrow 0,$$

where $\delta_a^{1*}(x)=-a_2^*x\oplus a_1^*x \ (x\in H)$ and $\delta_a^{0*}(x_1\oplus x_2)=a_1^*x_1+a_2^*x_2 \quad (x_1,x_2\in H)$. We recall that the pair $a^*=(a_1^*,a_2^*)$ is nonsingular on H if the sequence (2.1) is exact.

Lemma 2.1. If $a=(a_1,a_2)$ is nonsingular on H, then both $a_1^*a_1+a_2^*a_2$ and $a_1a_1^*+a_2a_2^*$ are invertible on H.

Proof. Let us show that $a_1^*a_1+a_2^*a_2$ is injective and surjective on H. If $(a_1^*a_1+a_2^*a_2)x=0$ for a certain $x\in H$, then $a_1x\oplus a_2x\in\ker\ \delta_a^{0*}=(\text{im}\ \delta_a^0)^\perp$. But $a_1x\oplus a_2x\in\text{im}\ \delta_a^0$; hence $a_1x\oplus a_2x\in(\text{im}\ \delta_a^0)^\perp\cap\text{im}\ \delta_a^0=\{0\}$. Thus $a_1x=a_2x=0$. Since $\ker\ \delta_a^0=0$, we have $x=0$. Take an arbitrary $y\in H$ and let us find an $x\in H$ such that $y=a_1^*a_1x+a_2^*a_2x$. We infer that $\delta_a^{0*}:(\ker\delta_a^{0*})^\perp\to H$ is an isomorphism, and therefore $y=\delta_a^{0*}(y_1\oplus y_2)$ with $y_1\oplus y_2\in(\ker\ \delta_a^{0*})^\perp=\text{im}\ \delta_a^0$; hence $y_1\oplus y_2=a_1x\oplus a_2x$. Analogously, the operator $a_1a_1^*+a_2a_2^*$ is invertible and this completes the proof of the lemma.

Theorem 2.2. Let $a=(a_1,a_2)\subset B(H)$ be a commuting pair. Then a is nonsingular on H if and only if the operator

$$(2.2)\qquad \alpha(a)=\begin{pmatrix} a_1^* & a_2^* \\ -a_2 & a_1 \end{pmatrix}$$

is invertible on $H\oplus H$.

Proof. According to Lemma 2.1, it is clear that the operator

$$(2.3)\qquad \begin{pmatrix} a_1(a_1^*a_1+a_2^*a_2)^{-1} & -a_2^*(a_1a_1^*+a_2a_2^*)^{-1} \\ a_2(a_1^*a_1+a_2^*a_2)^{-1} & a_2^*(a_1a_1^*+a_1a_2^*)^{-1} \end{pmatrix}$$

is a right inverse for the operator $\alpha(a)$ given by (2.2) ; hence $\alpha(a)$ is surjective on $H\oplus H$. Let us also notice that $\alpha(a)$ is injective too. Indeed, if $\alpha(a)(x_1\oplus x_2)=0$, then $x_1\oplus x_2\in\ker\ \delta_a^{0*}\cap\mathrm{im}\ \delta_a^1=\{0\}$, and hence $x_1=x_2=0$. Conversely, suppose that $\alpha(a)$ is invertible on $H\oplus H$. The $\alpha(a)^*$ is invertible ; therefore

$$\alpha(a)\alpha(a)^*=\begin{pmatrix} a_1^*a_1+a_2^*a_2 & 0 \\ 0 & a_1a_1^*+a_2a_2^* \end{pmatrix}$$

is invertible, and hence $(a_1^*a_1+a_2^*a_2)$ and $(a_1a_1^*+a_2a_2^*)$ are operators from $B(H)$. Let us prove that the sequence (1.1) is exact. Indeed, if $\delta_a^0(x)=a_1x\oplus a_2x=0$, then $(a_1^*a_1+a_2^*a_2)x=0$, whence $x=0$. Assume now that $\delta_a^1(x_1\oplus x_2)=a_1x_2-a_2x_1=0$. If $y=a_1^*x_1+a_2^*x_2$, then $\alpha(a)(x_1\oplus x_2)=y\oplus 0$; hence $x_1\oplus x_2=\alpha(a)^{-1}(y\oplus 0)$, and thus on account of (2.3) we obtain

$$x_1=a_1(a_1^*a_1+a_2^*a_2)^{-1}y,$$

$$x_2=a_2(a_1^*a_1+a_2^*a_2)^{-1}y,$$

i.e. the exactness of (1.1) at the second step. Finally, if $y\in H$ is arbitrary, then $x_j=a_j^*(a_1a_1^*+a_2a_2^*)^{-1}y$ $(j=1,2)$ satisfy the equation $a_1x_1+a_2x_2=y$, and the proof is complete.

Notice that $a=(a_1,a_2)\subset B(H)$ is nonsingular if and only if the matrix

$$\alpha(a^*)=\begin{pmatrix} a_1 & a^2 \\ -a_2^* & a_1^* \end{pmatrix}$$

is invertible on $H\oplus H$, and also if and only if the matrix

$$\alpha(a)^* = \begin{pmatrix} a_1 & -a_2^* \\ a_2 & a_1^* \end{pmatrix}$$

is invertible on $H\oplus H$.

Corollary 2.3. If A is any commutative algebra of operators on H, then the map

$$A^2 \ni a \longrightarrow \alpha(a) \in B(H\oplus H)$$

is R-linear.

Proof. Since the maps δ_a^0 and δ_a^1 are linear on A^2, $\alpha(a)$ is R-linear.

Remark. The set of matrices $\{\alpha(z) : z\in C^2\}$ can be identified with the algebra of quaternions and that the map $z \rightarrow \alpha(z)$ is an R-linear isometric isomorphism[10].

Corollary 2.4. For any $z=(z_1,z_2)\in C^2$, $z\neq 0$, $\alpha(z)^{-1}$ exists and $\alpha(z)^{-1} =(|z_1|^2+|z_2|^2)^{-1}\alpha(z)$.

Proof. It is easy to see that $\sigma(z,H)=\{z\}$, hence $\alpha(z)$ is invertible for any $z\neq 0$. Then

$$\alpha(z)^{-1}=(|z_1|^2+|z_2|^2)^{-1}\alpha(z).$$

Corollary 2.5. For any $z\in C^2$, $z\neq 0$, we have

$$\|\alpha(z)\| = \|z\| \quad \text{and} \quad \|\alpha(z)^{-1}\| = \|z\|^{-1}$$

where $\|z\|^2=|z_1|^2+|z_2|^2$.

Corollary 2.6. If $a=(a_1, a_2)$ is nonsingular on H, then we have the following commuting relations:

$$a_1(a_1^*a_1+a_2^*a_2)^{-1}a_1^*+a_2^*(a_1a_1^*+a_2a_2^*)^{-1}a_2=1$$

$$(2.4)\qquad a_2(a_1^*a_1+a_2^*a_2)^{-1}a_2^*+a_1^*(a_1a_1^*+a_2a_2^*)^{-1}a_1=1$$

$$a_1(a_1^*a_1+a_2^*a_2)^{-1}a_2^*+a_2^*(a_1a_1^*+a_2a_2^*)^{-1}a_1=0.$$

Formulas (2.4) can be obtained by using the fact that (2.3) provides also a left inverse for $\alpha(a)$.

3. Joint spectrum

Lemma 3.1. For a commuting pair of operators $a=(a_1,a_2)\subset B(H)$, we have $\sigma(a,H)=C^2-\{z\in C^2\,;\,(z-a)^{-1}\in B(H\oplus H)\}$.

Corollary 3.2. If $a=(a_1,a_2)\subset B(H)$ is a commuting pair, then $\alpha(a,H)=C^2-\{z\in C^2\,;\,(\alpha(z)-\alpha(a))^{-1}\in B(H\oplus H)\}$.

Definition 3.3. The mapping

$$C^2-\alpha(a,H)\ni z\longrightarrow R(z,a)=(\alpha(z)-\alpha(a))^{-1}\in B(H\oplus H)$$

is called the resolvent of a.

Lemma 3.4. For a commuting pair $a=(a_1,a_2)\subsetneq B(H)$, the joint spectrum $\sigma(a,H)$ is a closed set and the resolvent $R(z,a)$ is an R-analytic function in $C^2-\sigma(a,H)$.

Proof. Fix a point $z_0\notin\sigma(a,H)$. Since the map $z\to\alpha(z)$ is isometric, then for $z\in C^2$ such that $\|z-z_0\|<\|\alpha(z_0-a)-1\|^{-1}$, the series

$$\alpha(z_0-a)^{-1}\sum_{k=0}^{\infty}(-1)^k(\alpha(z-z_0)\alpha(z_0-a)^{-1})^k$$

is absolutely convergent and defines $(z-a)^{-1}$. In particular, the set $C^2-\sigma(a,H)$ is open. Notice that $\alpha(z-z_0)$ is a polynomial of degree one in z_l and $\dot{z}_k$, where we get easily that $\alpha(z-a)^{-1}$ is R-analytic in $C^2-\sigma(a,H)$.

Lemma 3.5. For a commuting pair $a=(a_1,a_2)\subset B(H)$ and any z in C^2 such that $\|z\|>\|\alpha(a)\|$, we have z is not in $\sigma(a,H)$ and

$$(3.1)\qquad (\alpha(z)-\alpha(a))^{-1}=\sum_{k=0}^{\infty}(\alpha(z)^{-1}\alpha(a))^k\alpha(z)^{-1}$$

is absolutely and uniformly convergent on the sets $\{z\in C^2 ; \|z\|>r\}$ with $r>\|\alpha(a)\|$.

Proof. According to Corollary 2.5, we have that if $\|z\|>\|\alpha(a)\|$, then $\|\alpha(a)^{-1}\alpha(a)\|<1$, hence the series (3.1) is absolutely convergent. It is straightforward to verify that (3.1) defines the inverse of $\alpha(a)$-$\alpha(a)$. If $r>\|\alpha(a)\|$, then for any z in C^2 such that $\|z\|>r$ we obtain by a direct estimation

$$\|(\alpha(z)-\alpha(a))^{-1}\|\leq r^{-1}(r-\|\alpha(a)\|)^{-1},$$

hence the convergence of (3.1) is uniform.

Notice that $\lim_{\|z\|\to\infty}\|(\alpha(z)-\alpha(a))^{-1}\|=0.$

Theorem 3.6. Let $a=(a_1,a_2)\subset B(H)$ be a pair of commuting operators. Then the joint spectrum $\sigma(a,H)$ of a is a compact nonempty set in C^2.

Proof. On account of the Lemma 3.4 and Lemma 3.5, $\sigma(a,H)$ is a compact subset of C^2. Let us assume that $\sigma(a,H)$ is empty. Then by Theorem 2.2 the operator

$$((z_1-a_1)^*(z_1-a_1)+(z_2-a_2)^*(z_2-a_2))^{-1}$$

does exist, therefore the right ideal generated in $B(H)$ by $z-a$ is equal to $B(H)$ for any $z=(z_1,z_2)\in C^2$ which is, according to [3], a contradiction.

4. Applications

Let H be a fixed Hilbert space. Let $a=(a_1,a_2)$ be a commuting pair of linear operators on H and let K be a closed subspace of H, K reducing a, i.e. $a_jK\subset K$, $a_j^*K\subset K$ for $j=1,2$. We denote by $a|K$ the restrictions $(a_1|K, a_2|K)$.

Proposition 4.1. Assume that $a=(a_1,a_2)\subset B(H)$ is nonsingular on H and K be a closed subspace of H, K reducing a. Then $a|K$ is nonsingular if and only if $\alpha(a)^{-1}(K\oplus K)\subset K\oplus K$.

Proof. We apply Theorem 2.2 If $a|K$ is nonsingular, then $\alpha(a|K)^{-1}\in B(K\oplus K)$. Take $\eta\in K\oplus K$. We have

$$\alpha(a)(\alpha(a)^{-1}\eta-\alpha(a|K)^{-1}\eta)=0,$$

hence $\alpha(a)^{-1}\eta=\alpha(a|K)^{-1}\eta\subset K\oplus K$. Conversely, if $\alpha(a)^{-1}(K\oplus K)\subset K\oplus K$, then we have

$$\alpha(a)^{-1}|K=\alpha(a|K)^{-1},$$

hence $a|K$ is nonsingular.

For any set $F\subset C^2$, let us denote by ∂F the boundary of F.

Proposition 4.2. Let K be a closed subspace of H, K reducing a. Then we have the relation

$$\partial\sigma(a,K)\subset\sigma(a,H).$$

Proof. Let us choose a point $z_0\in\sigma(a,K)$ and suppose that $z_0\notin\sigma(a,H)$.

Then there is a sequence $z_k \notin \sigma(a,K) \cup \sigma(a,H)$ such that $z_k \to z_0$ as $k \to \infty$. If $\eta \in K \oplus K$ is arbitrary, we can write

$$(\alpha(z_0) - \alpha(a))^{-1}\eta = \lim_{k \to \infty} (\alpha(z_k) - \alpha(a))^{-1}\eta \in K \oplus K,$$

therefore $\alpha(z_0) - \alpha(a)$ is nonsingular on K, which is a contradiction.

Corollary 4.3. If K_1 and K_2 are closed subspaces, reducing a, such that $\sigma(a,K_1) \cap \sigma(a,K_2) = \phi$, then $K_1 \cap K_2 = 0$,

Proof. Indeed, $K_1 \cap K_2$ is reducing a, therefore

$$\partial\sigma(a,K_1 \cap K_2) \subset \sigma(a,K_1) \cap \sigma(a,K_2) = \phi,$$

hence

$$\sigma(a,K_1 \cap K_2) = \phi, \text{ thus } K_1 \cap K_2 = 0.$$

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