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PETTIS DECOMPOSABLE OPERATORS AND THE BOURGAIN PROPERTY

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1. Introduction

In 1988, E.M. Bator [2] introduced a decomposition of bounded scalarly measurable functions taking their ranges in dual of a Banach space into a Pettis integrable part and weak* null part. And she extended Musial's result ([9], Theorem 5.3) to the case X not necessarily separable by a suitable weakening of the conclusion.

Using Bator's idea, we obtain *Odell's characterization.*

In 1982, L.H. Riddle [10] proved the following theorem :

Theorem : Let (Ω,Σ,μ) be a separable measure space. If $S : L_1(\mu)\rightarrow X^*$ be a bounded linear operator with the Bourgain property, then S is Pettis representable. And he asked whether the converse is true.

In this paper we define a new bounded linear operator on $L_1[0,1]$ which is called a *Pettis decomposable operator* and our main theorem gives a partial answer to the above question.

2. Preliminaries

Definition 2.1. A finite measure space (Ω,Σ,μ) is perfect if for each measurable map $f : \Omega\rightarrow R$ and each set $F\subseteq R$ for which $f^{-1}(F)\in\Sigma$, there is a Borel set $G\subseteq F$ with $\mu f^{-1}(G)=\mu f^{-1}(F)$.

Definition 2.2. A subset B of a Banach space X is called weakly precompact if every bounded sequence in B has a weakly Cauchy subsequence.

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Definition 2.3. Let $f:\Omega\to X^*$ be a weak* measurable function. f is said to have the RS-property if the Radon image measure $\upsilon=\mu\circ f^{-1}$ is such that for every n there is a Pettis set K_n such that $\upsilon(\Omega\setminus K_n)\leq\frac{1}{2^n}$.

Talagrand [12] showed that if f is weak* scalarly bounded, then f has the RS-property if and only if for every $\varepsilon>0$ there exists $E\in\Sigma$ with $\mu(\Omega\setminus E)<\varepsilon$ such that the set $\{\langle f,x\rangle\chi_E : \|x\|\leq 1\}$ is weakly precompact.

Definition 2.4. Let (Ω,Σ,μ) be a finite measure space. A family ψ of real-valued functions on Ω is said to have the Bourgain property if the following condition is satisfied : for each set A positive measure and for each $\alpha>0$, there is a finite collection F of subsets of A of positive measure such that for each function f in ψ, the inequality sup f(B)-inf f(B) $<\alpha$ holds for some member B of F.

The next theorem is due to Bourgain [5].

Theorem 2.5. If (Ω,Σ,μ) is a finite measure space and ψ is a family of real-valued functions on Ω satisfying the Bourgain property, then

(i) the pointwise closure of ψ satisfies the Bourgain property,

(ii) each element in the pointwise closure of ψ is measurable, and

(iii) each element in the pointwise closure of ψ is the almost everywhere pointwise limit of a sequence from ψ.

It is worth remarking here that a uniformly bounded family ψ of real-valued functions has the Bourgain property if and only if the following condition holds :

For each non-null measurable set A in Σ and for each pair $a<b$ of real numbers, there is a finite collection F of non-null measurable subsets of A such that for each f in ψ, either inf f(B)$\geq$a or sup

$f(B) \leq b$ for some member B of F.

We shall say that f has the Bourgain property if the family $\{\langle f,x\rangle : \|x\| \leq 1\}$ has the Bourgain property.

Theorem 2.6. A bounded function $f : \Omega \to X^*$ that has the Bourgain property is Pettis integrable.

The following theorem is due to Bator [1].

Theorem 2.7. Let X be a Banach space and (Ω,Σ,μ) a finite measure space. Suppose $f : \Omega \to X^*$ is bounded and weakly measurable. Then f is Pettis integrable if and only if, for every $x^{**} \in X^{**}$, there exists a bounded sequence $(x_n)_{n=1}^{\infty}$ in X such that both of the following hold :

(i) $\hat{x}_n \circ f$ converes a.e. to $x^{**} \circ f$,

(ii) $\hat{x}_n((w^*)\text{-}\int_E f\, d\mu)$ converges to $x^{**}((w^*)\text{-}\int_E f\, d\mu)$ for every $E \in \Sigma$.

3. Pettis decomposition and the weak Radon-Nikodym property

Rosenthal gave E. Odell's characterization of those spaces X not containing ℓ_1 : The Banach space X fails to contain an isomorphic copy ℓ_1 if and only if every Dunford-Pettis operator $T : X \to Y$ is compact for every space Y.

In this section, combining Lemma 3.4 and Corollary 3.6, we obtain Odell's characterization.

Definition 3.1. Let X be a Banach space and (Ω,Σ,μ) be a finite measure space and $f : \Omega \to X^*$ be bounded scalarly measurable. f is called Pettis decomposable if there exists a Pettis integrable function g and a weak*-null function h such that $f = g + h$.

The following Proposition is in [2].

Proposition 3.2. Let X be a Banach space and (Ω,Σ,μ) be a finite measure space. If f is a bounded and scalarly measurable then the following are equivalent :

(i) There exists a μ-Pettis integrable function g and a μ-weak* null function h such that $f=g+h$.

(ii) There exists a μ-Pettis integrable function g such that for every $x^{**}\in X^{**}$, $T_f^{**}(x^{**})=x^{**}\circ g$ in $L_1(\mu)$.

(iii) For every $\varepsilon>0$, there exists $A\in\Sigma$ and a Pettis integrable function g such that $\mu(\Omega\setminus A)<\varepsilon$ and $(x\circ f)\chi_A=x\circ g$ *a.e.*-$[\mu]$ for every $x\in X$.

The following Corollary is obvious.

Corollary 3.3. Let X be a separable Banach space and (Ω,Σ,μ) be a finite measure space. If $f:\Omega\to X^*$ be a bounded scalarly measurable, the following are equivalent :

(i) f is Pettis integrable,

(ii) f is Pettis decomposable.

Lemma 3.4. Let (Ω,Σ,μ) be a perfect measure space and $f:\Omega\to X^*$ be a bounded weak* scalarly measurable function. If $f=g+h$, where g is scalarly measurable and h is weak* null, then the operator $T_f: X\to L_1(\mu)$, defined by $T_f(x)=x\circ f$ for every $x\in X$, is compact.

Proof. Since h is weak* null, $T_f(x)=T_g(x)$ in $L_1(\mu)$ for every $x\in X$. However, since g is scalarly measurable, the operator T_g is compact by Proposition 3 of [1].

The following theorem is the main theorem of [2] which is the extension of Musial's result [9].

Theorem 3.5. If X is a Banach space, then the following are equivalent :

(i) X does not contain an isomorphic copy of ℓ_1.

(ii) X^* has the WRNP.

(iii) If (Ω,Σ,μ) is a complete measure space and $f:\Omega\to X^*$ is bounded and weak* scalarly measurable, then f is Pettis decomposable.

(iv) If (Ω,Σ,μ) is a complete measure space and $f:\Omega\to X^*$ is bounded weak* scalarly measurable, then $f=g+h$, where g is scalarly measurable and h is weak* null.

As a corollary of the above theorem, we obtain the following result.

Corollary 3.6. If X is a Banach space, then the following statements concerning X are equivalent:

(i) X^* has the WRNP.

(ii) Given any complete measure space (Ω,Σ,μ) and any bounded weak* scalarly measurable function $f:\Omega\to X^*$, f is weak* equivalent to a scalarly measurable function.

(iii) Given any bounded weak* scalarly measurable function $f:[0,1]\to X^*$ on the unit interval endowed with the Lebesgue measurable sets and the Lebesgue measure, f is weak* equivalent to a scalarly measurable function.

(iv) Given any complete measure space (Ω,Σ,μ) and any bounded weak* scalarly measurable function $f:\Omega\to X^*$, f is weak* equivalent to a Pettis integrable function.

(v) Given any bounded weak* scalarly measurable function $f:[0,1]\to X^*$ with the unit interval endowed with the Lebesgue measurable sets and the Lebesgue measure, f is weak* equivalent to a Pettis integrable function.

(vi) X does not contain any isomorphic copy of ℓ_1.

Proof. Using the same arguments as in the proof of Theorem 3.5, we see that the implications (iv)→(ii)→(iii) and (iv)→(i)→(iv)→(v)→(iii) hold. Janika proved the equivalence of (i) and (vi) in [8]. We have only left to show (iii) implies (vi). If X contains ℓ_1, then there exists a bounded weak* Lebesgue measurable function $f:[0,1]$

$\to X^*$ such that the operator $T_f : X \to L_1[0,1]$ is not compact[4]. Let $f : [0,1] \to X^*$ be weak* equivalent to g which is scalarly measurable. Then clearly $x \circ f = x \circ g$ for every $x \in X$. Thus $T_f(x) = T_g(x)$ in $L_1(\mu)$ for every $x \in X$ and T_f is compact by Proposition 3 of [1]. This completes the proof.

Now, we obtain Odell's characterization.

Theorem 3.7. If X is a Banach space, then the following are equivalent:

(i) X does not contain an isomorphic copy of ℓ_1.

(ii) Every Dunford-Pettis operator $T : X \to Y$ is compact for every space Y.

Proof. (i)→(ii): Suppose that X does not contain a copy of ℓ_1, $T : X \to Y$ is D-P, and (x_n) is a bounded sequence. Then (x_n) has a weakly Cauchy subsequence, say $(x_{n_i})_{i=1}^{\infty}$ Consequently $(T(x_{n_i}))$ is norm convergent, and hence T is compact.

(ii)→(i): Suppose that X contains an isomorphic copy of ℓ_1. Then by Corollary 3.6 and Lemma 3.4, there exists a bounded weak*-Lebesgue measurable function $f : [0,1] \to X^*$ such that $T_f : X \to L_1[0,1]$ is not compact. Let(π_n) be a sequence of the dyadic partitions of $[0,1]$ and Σ_n denotes the σ-algebra generated by π_n. Define an X*-valued martingale (f_n, Σ_n) by

$$f_n = \sum_{A \in \pi_n} \frac{(w^*) - \int_A f d\mu}{\mu(A)} \chi_A.$$

Then $(\langle f_n, x \rangle, \Sigma_n)$ is a uniformly bounded martingale with $\lim_n \langle f_n, x \rangle = \langle f, x \rangle$ a.e. and hence in $L_1[0,1]$. Hence by Bounded Convergence Theorem T_f is a D-P operator and completes the proof.

4. Pettis decomposable operators and the Bourgain property

In this section, using Bator's idea, we define a bounded linear operator $S: L_1[0,1] \to X^*$ which is called a Pettis decomposable operator. The symbol $L_1[0,1]$ represents the space $L_1([0,1],\Sigma,\mu)$ where Σ is the σ-algebra of Lebesgue measurable subsets of $[0,1]$ and μ is the Lebesgue measure.
Let (π_n) be a sequence of the dyadic partitions of $[0,1]$ and Σ_n denotes the σ-algebra generated by π_n. Let $S: L_1[0,1] \to X^*$ be a bounded linear operator. For each $n \in N$ define a function $f_n: [0,1] \to X^*$ by

$$f_n = \sum_{A \in \pi_n} \frac{S(\chi_A)}{\mu(A)} \chi_A.$$

Then the sequence (f_n,Σ_n) forms a uniformly bounded X^*-valued martingale. We shall say that the sequence (f_n,Σ_n) is the associated martingale with S.

Definition 4.1. Let $S: L_1[0,1] \to X^*$ be a bounded linear operator with the associated martingale (f_n,Σ_n). The operator S has the Bourgain property if the family $\{\langle f_n,x\rangle : n \in N, \|x\| \leq 1\}$ has the Bourgain property.

Definition 4.2. Let $S: L_1[0,1] \to X^*$ be a bounded linear operator with the associated martingale (f_n). The operator S is called Pettis decomposable if there exists a pointwise weak*-cluster point $f: [0,1] \to X^*$ of $\{\langle f_n,x\rangle : n \in N, \|x\| \leq 1\}$ such that f is Pettis decomposable.

Theorem 4.3. An operator $S: L_1[0,1] \to X^*$ with the Bourgain property is Pettis decomposable.

Proof. Let (f_n) be the uniformly bounded X^*-valued martingale associated with S. Choose a pointwise weak*-cluster point $f: [0,1] \to X^*$

of (f_n). Let $x \in B_X$. Then $\lim \langle f_n,x\rangle = \langle f,x\rangle$ a.e.. By the Bounded Convergence Theorem

$$S(g)x = \int \langle f,x\rangle g \; d\mu \quad for \;\; all \;\; g \in L_1[0,1].$$

Since $\{\langle f,x\rangle : \|x\| \leq 1\}$ lies in the pointwise closure of $\{\langle f_n,x\rangle : n \in N, \|x\| \leq 1\}$, f has the Bourgain property. So f is Pettis integrable by Theorem 2.6. Clearly f is scalarly measurable and bounded. Hence f is Pettis decomposable and therefore S is Pettis decomposable.

In [2], Bator proved the following theorem :

Theorem 4.4. Let (Ω,Σ,μ) be a finite measure space and X a Banach space. If $f : \Omega \rightarrow X^*$ is a bounded scalarly measurable function such that f has the RS-property, then f is Pettis decomposable.

Theorem 4.5. Let $S : L_1[0,1] \rightarrow X^*$ be a bounded linear operator with the associated martingale (f_n) and f be a pointwise weak*-cluster point of (f_n). If f be a scalarly measurable function having the RS-property, then S is Pettis decomposable.

Proof. If f has the RS-property, then there exists a Pettis integrable function $g : [0,1] \rightarrow X^*$ such that $x \circ f = x \circ g$ in $L_1[0,1]$ for all $x \in B_X$[12]. Thus by Proposition 3.2, f is Pettis decomposable. Hence S is Pettis decomposable.

Theorem 4.6. Let $S : L_1[0,1] \rightarrow X^*$ be a Pettis decomposable operator and X a separable Banach space. Then S is Pettis representable.

Proof. Let $S : L_1[0,1] \rightarrow X^*$ be a bounded linear operator with the associated martingale (f_n). Since S is Pettis decomposable, there exists a pointwise weak*-cluster point $f : [0,1] \rightarrow X^*$ such that f is a Pettis decomposable. Let $x \in B_X$. Then $S(g)x = \int \langle f,x\rangle g \; d\mu$ for all $g \in L_1[0,1]$. Since X is separable, f is Pettis integrable.

Let $x^{**}\in X^{**}$. Then by Theorem 2.7 and the Bounded Convergence Theorem,

$$\langle x^{**},S(g)\rangle=\lim_n\langle x_n,S(g)\rangle=\lim_n\langle f,x_n\rangle g\ d\mu$$

$$=\int\langle f,\ \lim_n x_n\rangle d\mu=\int\langle x^{**},f\rangle g\ d\mu$$

Thus f is a Pettis derivative of S and hence S is Pettis representable.

The following Lemma is in [5].

Lemma 4.7 (Bourgain) Suppose A is a subset of [0,1] with positive measure and $0<\alpha<1$. Then there is an integer m and a measurable subset $B\subseteq A$ with $\mu(B)>(1-\alpha)\mu(A)$ such that for every uniformly bounded by 1 real-valued martingale (g_n,Σ_n) and for every $n\geq m$,

(i) ess inf $g(A)\leq$ ing $g_n(B)+\alpha$

(ii) ess sup $g(A)\geq\sup g_n(B)-\alpha$

where g is any almost everywhere limit of the sequence (g_n).

The following Lemma is need to our main theorem in this section.

Lemma 4.8. Let $S: L_1[0,1]\rightarrow X^*$ be a bounded linear operator with the associated martingale (f_n) and $f: [0,1]\rightarrow X^*$ be a pointwise weak*-cluster point of (f_n).

If f has the Bourgain property, then S has the Bourgain property.

Proof. Without loss of generality, we may assume that $\|S\|\leq 1$ Suppose that the family $\{\langle f,x\rangle : \|x\|\leq 1\}$ has the Bourgain property. Let A be a set of positive measure and a<b, Choose $\alpha>0$ such that $a+\alpha<b-\alpha$. There exists $A_1,\ldots,A_k$ of with positive measures such that for each $x\in B_X$, either $\sup_{A_i}\langle f,x\rangle\leq b-\alpha$ or $\inf_{A_i}\langle f,x\rangle\geq a+\alpha$ for some i. Since $\lim_n\langle f_n,x\rangle=\langle f,x\rangle$ a.e., according to Lemma 4.7, there exists, for each set A_i, an integer m_i and a non-null subset of B_i of

A_i such that

(i) $\text{ess} \inf_{A_i} \langle f,x\rangle \leq \inf_{B_i} \langle f_n,x\rangle + \alpha$

(ii) $\text{ess} \sup_{A_i} \langle f,x\rangle \leq \sup_{B_i} \langle f_n,x\rangle - \alpha$ for each $x \in B_X$ and for every $n \geq m_i$.

Let $m = max\{m_i : 1 \leq i < k\}$. Let $n \geq m$, let $x \in B_X$, and note that there exists an A_i such that either

$$b-\alpha \geq \sup_{A_i} \langle f,x\rangle \leq \text{ess} \sup_{A_i} \langle f,x\rangle \leq \sup_{B_i} \langle f_n,x\rangle - \alpha$$

or

$$a+\alpha \leq \inf_{A_i} \langle f,x\rangle \leq \text{ess} \inf_{A_i} \langle f,x\rangle \leq \inf_{B_i} \langle f_n,x\rangle - \alpha$$

That is, either $b \geq \sup_{B_i} \langle f_n,x\rangle$ or $a \leq \inf_{B_i} \langle f_n,x\rangle$. Therefore the sets $B_1, \ldots, B_k$ will work for the set A for the family $\{\langle f_n,x\rangle : n \geq m, \|x\| \leq 1\}$. However the $f_1, \ldots f_{m-1}$ are just simple, so that for each $i = 1, \ldots, m-1$ there exists a set C_i on which f_i is constant and $\mu(A \cap C_i) > 0$, Thus the sets $B_1, \ldots B_k, C_1 \cap A, \ldots, C_{m-1} \cap A$ will work for the set A for the family $\{\langle f_n,x\rangle : n \in N, \|x\| \leq 1\}$. Thus S has the Bourgain property.

Theorem 4.9. Let $S : L_1[0,1] \to X^*$ be a Pettis decomposable operator with the associated martingale (f_n) and with a Pettis decomposable function $f : [0,1] \to X^*$. If $f = g + h$, where g has the Bourgain property and h is weak*-null, then S has the Bourgain property.

Proof. Let $f : [0,1] \to X^*$ be a pointwise weak*-cluster point of (f_n) and $f = g + h$, where g has the Bourgain property and h is weak*-unll. Let $x \in B_x$. Then $\langle f,x\rangle$ is a pointwise cluster point of the sequence $(\langle f_n,x\rangle)$. We must show that $\{\langle f_n,x\rangle : n \in N, \|x\| \leq 1\}$ has the Bourgain property. Suppose that $\{\langle f_n,x\rangle : n \in N, \|x\| \leq 1\}$ fails the Bourgain property. Then by Theorem 4.8 f does not have the Bourgain property. However since f-g is weak*-null, g does not have the Bourgain property. This is a contradiction and completes the proof.

References

1. E.M.Bator, The Pettis Integral and the equality of the norms of the Dunford integral and the weak* integral, Proc. Amer. Math. Soc. 95(1985), 265−270.
2. ———, A decomposition of bounded scalarly measurable functions taking their ranges in dual Banach spaces, Proc. Amer. Math. Soc. 102(1988), 850−854.
3. ———, Pettis decomposition for universally scalarly measurable functions, Proc. Amer. Math. Soc. 104(1988), 795−800.
4. E.M. Bator, P.W. Lewis, and D.Race, Some connections between Pettis integration and operator theory, Rocky Mountain Math. 17(1987), 683−695.
5. J.Bourgain, Martingales in conjugate Banach spaces (preprint).
6. J.Diestel and J.J.Uhl, Jr., Vector measures, Math. Surveys, no. 15, Amer. Math. Soc., Providence, R.I., 1977.
7. N.Dunford and J.T. Schwartz, Linear operators Part I, Interscience, New York, 1958.
8. L.Janika, Some measure-theoretic characterizations of Banach spaces not containing ℓ_1, Bull. Acad. Polon. Sci. 27(1979), 561−565.
9. K.Musial, The weak Radon-Nikodym property in Banach spaces, Studia Math. 641(1978), 151−174.
10. L.H.Riddle, Weak Radon-Nikodym sets in dual Banach spaces, PH.D. Thesis, University of Illinois, 1982.
11. H.P. Rosenthal, A characterization of Banach spaces containing ℓ_1, Proc. Nat. Acad. Sci. U.S.A. 71(1974), 2411−2413.
12. M.Talagrand, Pettis integral and measure theory, Mem. Amer. Math. Soc., No. 307(1984).

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