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ZARANTONELLO TYPE INEQUALITIES FOR LIPSCHITZIAN MAPPINGS IN BANACH SPACES

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1. Introduction.

Throughout this paper X denotes a uniformly convex Banach space and C is a bounded closed convex nonempty subset of X. A mapping $T: C \to X$ is called a lipschitzian mapping if there exists a positive number k such that

$$\| Tx - Ty \| \leq k \| x - y \| \quad \text{for every } x, y \varepsilon C,$$

and especially nonexpansive in the case of $k=1$.

Zarantonello's inequality [1] is valid in Hibert spaces as fallows : Let H be a real Hilbert space and C be a bounded closed nonempty convex subset of H. If T is a contractive self-mapping of C, then for all $x_i \varepsilon C$ and $\lambda_i \geq 0$, $i=1,2,\ldots n$ with $\sum_{i=1}^{n} \lambda_i = 1$,

$$\| T(\sum_{i=1}^{n} \lambda_i x_i) - \sum_{i=1}^{n} \lambda_i T x_i \|^2 \leq \sum_{1 \leq i \leq j \leq n} \lambda_i \lambda_j (\| x_i - x_j \|^2 - \| Tx_i - Tx_j \|^2)$$

Miyazaki[2] provided an analogue of Zarantonello's inequality in uniformly convex Banach spaces with nonexpansive mapping.

The purpose of this paper is to provide an analogue of Zaranotonello's inequality in uniformly convex Banach spaces with lipschitzian mapping.

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First we introduce the definition of lipschitzian mappings of strong type(γ) and give a condition of lipschitzian mapping of strong type(γ) and second we prove an analogue of Zarantonello's inequality in Banach spaces with lipschitzian mappings.

2. Lipschitzian mappings of strong type(γ)

We denote by Γ the set of strictly increasing convex, hence continuous functions $\gamma : R^+ \to R^+$ with $\gamma(0)=0$. In [2], a mapping $T : C \to X$ is said to be strong type(γ) if $\gamma \varepsilon \Gamma$ and for all x,y in C and any c, $0 \le c \le 1$,

$$\gamma(\| cTx+(1-c)Ty-T(cx+(1-c)y) \|) \\ \le c(1-c)(\| x-y \| - \| Tx-Ty \|).$$

Obviously, every mappings of strong type(γ) is both a mapping of type(γ) and a contraction. But, not every contraction is of strong type(γ).

Definition. Let X be a Banach space, C be a bounded closed convex subset of X. A mapping $T : C \to X$ is said to be lipschitzian strong type(γ) with lipschitzian constant k if $\gamma \varepsilon \Gamma$ and for all x,y in C and any λ, $0 \le \lambda \le 1$,

$$\gamma(\frac{1}{k} \| \lambda Tx+(1-\lambda)Ty-T(\lambda x+(1-\lambda)y) \|) \\ \le \lambda(1-\lambda)(\| x-y \| - \frac{1}{k} \| Tx-Ty \|) \tag{1}$$

Theorem 2.1. If X is a uniformly convex Banach space and C be a bounded closed convex subset of X, then there exists a $\gamma \varepsilon \Gamma$ such that every lipschitzian mapping $T : C \to X$ is of lipschitzian strong type(γ) with lipschitzian constant k.

Proof. Let δ be the modulus of uniform convexity of X:

$$\delta(t)=\inf\{1-\frac{\|u+v\|}{2} \mid \|u\|\leq 1, \|v\|\leq 1, \quad \|u-v\|\geq t\}.$$

Then $\delta:[0,2]\to[0,1]$ is continuous, increasing, $\delta(0)=0$, and $\delta(t)>0$ for $t>0$, while

$$2\ \min(\lambda,1-\lambda)\delta(\|u-v\|)\leq 1-\|\lambda u-(1-\lambda)v\| \tag{2}$$

whenever $0\leq\lambda\leq 1$ and $\|u\|\leq 1$, $\|v\|\leq 1$.

Now we put

$$d_1(t)=\begin{cases}\frac{1}{2}\int_0^t \delta(s)\ ds & \text{for } 0\leq t\leq 2\\ d_1(2)+\delta(2)(t-2) & \text{for } t>2,\end{cases}$$

and

$$d_2(t)=\frac{1}{2}\int_0^t d_1(s)\ ds,$$

then it is shown that $d_1(t)\varepsilon\Gamma$, $d_2(t)\varepsilon\Gamma$, and $d_2(t)\leq d_1(t)\leq\delta(t)$ for $0\leq t\leq 2$ and $d_2(t)$ is two times differentiable and $\frac{d_2(t)}{t^2}$ is increasing for $t>0$. From (2) we have

$$2\lambda(1-\lambda)\ d_2(\|u-v\|)\leq 1-\|\lambda u+(1-\lambda)v\| \tag{3}$$

whenever $0\leq\lambda\leq 1$ and $\|u\|\leq 1$, $\|v\|\leq 1$. Since it is sufficient to prove (1) for $0<\lambda<1$ and $x\neq y$, taking $u=\{Ty-T(\lambda x+(1-\lambda)y)\}/\{\lambda k\|x-y\|\}$, $v=\{T(\lambda x+(1-\lambda)y)-Tx\}/\{(1-\lambda)k\|x-y\|\}$, we have from (3)

$$2\lambda(1-\lambda)\ d_2(\frac{\|\lambda Tx+(1-\lambda)Ty-T(\lambda x+(1-\lambda)y)\|}{\lambda(1-\lambda)k\|x-y\|})$$

$$\leq 1-\frac{\| Tx-Ty \|}{k \| x-y \|}.$$

Hence, let M denote the diameter of C and noting that $\lambda(1-\lambda)\ k \| x-y \| \leq \frac{kM}{4}$ and $\frac{d_2(t)}{t^2}$ is increasing, we get

$$\frac{M}{8} d_2(\frac{4}{kM} \| \lambda Tx+(1-\lambda)Ty-T(\lambda x+(1-\lambda)y) \|)$$

$$\leq \lambda(1-\lambda)(\| x-y \| -\frac{1}{k} \| Tx-Ty \|).$$

Thus, defining $\gamma(t)=\frac{M}{8} d_2(\frac{4}{M} t)$, we get (1). This completes the proof.

3. Zarantonello type inequalities for lipschitzian mappings

We state here an analogue of Zarantonello's inequality proved by Bruck.

Lemma(Bruck[4]). Let X be a uniformly convex Banach space, C be a bounded closed convex subset of X. If $T: C \to X$ is a contraction, then for all $x_i \varepsilon C$, $\lambda_i \geq 0$, $i=1,2,\ldots,n$ with $\sum_{i=1}^{n} \lambda_i = 1$ there exists a strictly increasing convex, continuous function $\gamma: R^+ \to R^+$ which is dependent of n and $\gamma(0)=0$, such that

$$\gamma(\| T(\sum_{i=1}^{n} \lambda_i x_i)-\sum_{i=1}^{n} \lambda_i Tx_i \|) \leq \max_{1 \leq i \leq j \leq n} (\| x_i-x_j \| - \| Tx_i-Tx_j \|).$$

Following Bruck's way, we now show a variant of this lemma for mappings which are not necessarily nonexpansive.

Theorem 3.1. Let X be a uniformly convex Banach space, C be a bounded closed nonempty convex subset of X. If $T: C \to X$ is a lipschitzian mapping with lipschitzian constant k, then there exists a $\gamma_n \varepsilon \Gamma$ dependent on $n \geq 2$ such that for all $x_i \varepsilon C$ and $\lambda_i \geq 0$, $i=1,2,\ldots,n$ with $\sum_{i=1}^{n} \lambda_i = 1$,

$$\gamma_n(\frac{1}{k} \| T(\sum_{i=1}^{n} \lambda_i x_i)-\sum_{i=1}^{n} \lambda_i Tx_i \|) \leq \max_{1 \leq i \leq j \leq n} (\| x_i-x_j \| -\frac{1}{k} \| Tx_i-Tx_j \|) \quad (4)$$

Proof. We shall prove the assertion by induction. We begin by setting $\gamma_2=\gamma$. Supposing that the assertion is true for any k elements in C and with some $\gamma_k \varepsilon \Gamma$, $k<n$, we shall prove (4) by induction on n. We put

$$u_j=(1-\lambda_n)x_j+\lambda_n x_n,$$
$$u_j'=(1-\lambda_n)Tx_j+\lambda_n Tx_n,$$

and $\mu_j=\dfrac{\lambda_j}{1-\lambda_n}$ for $j=1,2,\ldots,n-1$. Then $u_j \varepsilon C$, $\mu_j \geq 0$ for $j=1,2,\ldots,n-1$, $\sum_{i=1}^{n-1}\mu_i=1$, $\sum_{i=1}^{n}\lambda_i x_i=\sum_{j=1}^{n-1}\mu_j u_j$, and $\sum_{i=1}^{n}\lambda_i Tx_i=\sum_{j=1}^{n-1}\mu_j u_j'$. We lay out the computations as follows:

$$\frac{1}{k}\| T(\sum_{i=1}^{n}\lambda_i x_i)-\sum_{i=1}^{n}\lambda_i Tx_i \| =\frac{1}{k}\| T(\sum_{j=1}^{n-1}\mu_j u_j)-\sum_{j=1}^{n-1}\mu_j u_j' \|$$
$$\leq \frac{1}{k}\| T(\sum_{j=1}^{n-1}\mu_j u_j)-\sum_{j=1}^{n-1}\mu_j Tu_j \|$$
$$+\frac{1}{k}\sum_{j=1}^{n-1}\mu_j \| Tu_j-u_j' \| . \tag{5}$$

$$\gamma_{n-1}(\frac{1}{k}\| T(\sum_{j=1}^{n-1}\mu_j u_j)-\sum_{j=1}^{n-1}\mu_j Tu_j \|)\leq \max_{1\leq j\leq k\leq n-1} \| u_j-u_k \|$$
$$-\frac{1}{k}\| Tu_j-Tu_k \|). \tag{6}$$

$$\| u_j-u_k \| -\frac{1}{k}\| Tu_j-Tu_k \| \leq \| u_j-u_k \| -\frac{1}{k}\| u_j'-u_k' \| +\frac{1}{k}\| u_k'$$
$$-Tu_k \| +\frac{1}{k}\| u_j'-Tu_j \| . \tag{7}$$

$$\gamma_2(\frac{1}{k}\| Tu_j-u_j' \|)\leq \| x_j-x_n \| -\frac{1}{k}\| Tx_j-Tx_n \| \tag{8}$$

$$\| u_j-u_k \| -\frac{1}{k}\| u_j'-u_k' \| \leq \| x_j-x_k \| -\frac{1}{k}\| Tx_j-Tx_k \| \tag{9}$$

Put : $t=max\{ \| x_i-x_k \| -\frac{1}{k} \| Tx_i-Tx_k \| \ ; 1 \le i \le k \le n\}$.

Then by (8)

$$\frac{1}{k} \| Tu_i - u_i' \| \le \gamma_2^{-1}(t),$$

which combined with (9) and used in (7) yields

$$\begin{aligned} \| u_j-u_k \| -\frac{1}{k} \| Tu_j-Tu_k \| &\le \| u_j-u_k \| -\frac{1}{k} \| u_j'-u_k' \| +\frac{1}{k} \| u_k'-Tu_k \| +\frac{1}{k} \| u_j'-Tu_j \| \\ &\le \| x_j-x_k \| -\frac{1}{k} \| Tx_j-Tx_k \| +\gamma_2^{-1}(t) +\gamma_2^{-1}(t) \\ &\le t+2\gamma_2^{-1}(t). \end{aligned} \tag{10}$$

When used in (6) this yields

$$\frac{1}{k} \| T(\sum_{j=1}^{n-1} \mu_j u_j) - \sum_{j=1}^{n-1} \mu_j Tu_j \| \le \gamma_{n-1}^{-1}(t+2\gamma_2^{-1}(t)). \tag{11}$$

Finally, when (11) is used with (5) we get

$$\begin{aligned} \frac{1}{k} \| T(\sum_{j=1}^{n} \lambda_j x_j) - \sum_{j=1}^{n} \lambda_j Tx_j \| &\le \frac{1}{k} \| T(\sum_{j=1}^{n-1} \mu_j u_j) - \sum_{j=1}^{n-1} \mu_j Tu_j \| +\frac{1}{k} \sum_{j=1}^{n-1} \mu_j \| Tu_j-u_j' \| \\ &\le \gamma_{n-1}^{-1}(t+2\gamma_2^{-1}(t)) + \frac{1}{k}\gamma_2^{-1}(t). \end{aligned}$$

We define $\gamma_n^{-1}(t)=\gamma_{n-1}^{-1}(t+2\gamma_2^{-1}(t))+\frac{1}{k}\gamma_2^{-1}(t)$. Hence

$$\gamma_n(\frac{1}{k} \| T(\sum_{i=1}^{n} \lambda_i x_i) - \sum_{i=1}^{n} \lambda_i Tx_i \| \le \max_{1 \le i \le j \le n} (\| x_i-x_j \| -\frac{1}{k} \| Tx_i - Tx_j \|).$$

We give another proof of the following corollary which was proved by Hirano [3].

Corollary 3.1. Let C be a bounded closed convex subset of X and $T: C \to C$ be a nonexpansive mapping. Let $x \varepsilon C$, $f \varepsilon F(T)$. and $0 < \alpha \leq \beta < 1$. Then for each $\varepsilon > 0$, there exists $N > 0$ such that for all $n \geq N$,

$$\| T^k(\lambda T^n x + (1-\lambda)f) - (\lambda T^{n+k} x + (1-\lambda)f) \| < \varepsilon$$

for all $k > 0$ and $\lambda: \alpha \leq \lambda \leq \beta$.

Proof. Since $T^k: C \to C$ is a nonexpansive mapping, by Theorem 3.1, there exists $\gamma \varepsilon \Gamma$ such that

$$\begin{aligned} &\| \lambda T^k y + (1-\lambda) T^k z - T^k(\lambda y + (1-\lambda) z) \| \\ &\qquad \leq \gamma^{-1}(\| y - z \| - \| T^k y - T^k z \|) \end{aligned}$$

for $y, z \varepsilon C$, $0 \leq \lambda \leq 1$. Therefore taking $y = T^n x$ and $z = f$, we have

$$\begin{aligned} &\| \lambda T^k T^n x + (1-\lambda) f - T^k(\lambda T^n x + (1-\lambda) f) \| \\ &\qquad \leq \gamma^{-1}(\| T^n x - f \| - \| T^{n+k} x - f \|). \end{aligned}$$

Since the sequence $\{ \| T^n x - f \| \}_{n \varepsilon N}$ is decreasing, $\lim_{n \to \infty} \| T^n x - f \|$ exists and for any $\varepsilon > 0$ there exists a positive integers N such that for any integer $n \geq N$,

$$\| T^n x - f \| - \| T^{n+k} x - f \| < \varepsilon.$$

It follows

$$\| \lambda T^{n+k} x + (1-\lambda) f - T^k(\lambda T^n x + (1-\lambda) f) \| < \gamma^{-1}(\varepsilon).$$

This completes the proof.

Theorem 3.2. Let X be a Banach space, C be a bounded closed nonempty convex subset of X. If $T: C \to X$ is of lipschitzian strong type (γ) with lipschitzian constant k for some $\gamma \varepsilon \Gamma$, then there exists a $\gamma_n \varepsilon \Gamma$ dependent on $n \geq 2$ such that for all $x_i \varepsilon C$ and $\lambda_i \geq 0$, $i=1, 2, \ldots n$, with $\sum_{i=1}^{n} \lambda_i = 1$,

$$\gamma_n\left(\frac{1}{k}\| T(\sum_{i=1}^{n}\lambda_i x_i) - \sum_{i=1}^{n}\lambda_i Tx_i \|\right) \leq \sum_{1 \leq i \leq j \leq n} \lambda_i\lambda_j\left(\| x_i - x_j \| - \frac{1}{k}\| Tx_i - Tx_j \|\right) \tag{12}$$

Proof. We shall prove the assertion by induction. The assumption of T being of lipschitzian strong type (γ) shows that (12) is true for $n=2$. We now here note that the γ in Theorem 2.1 and the γ_n constructed later inductively starting with γ are two times differentiable. Thus we may assume for convience of caculation that any function of Γ is two times differentiable. Supposing that the assertion is true for any k elements in C and with some $\gamma_k \varepsilon \Gamma$, $k<n$, we shall prove (12) by induction on n. Since there exists at least one $\lambda_i < \frac{1}{2}$ we may assume $0 \leq \lambda_n \leq \frac{1}{2}$. We define $u_j = (1-\lambda_n)x_j + \lambda_n x_n$, $u_j' = (1-\lambda_n)Tx_j + \lambda_n Tx_n$ and $\mu_j = \lambda_j / 1-\lambda_n$ for $j=1,2,\ldots,n-1$. Then $u_j \varepsilon C$, $\mu_j \geq 0$ for $j=1,2,\ldots, n-1$, $\sum_{j=1}^{n-1}\mu_j = 1$ and $\sum_{i=1}^{n}\lambda_i x_i = \sum_{j=1}^{n-1}\mu_j u_j$, $\sum_{i=1}^{n}\lambda_i Tx_i = \sum_{j=1}^{n-1}\mu_j u_j'$. Now we have

$$\begin{aligned}\|\frac{1}{k}(T(\sum_{i=1}^{n}\lambda_i x_i) - \sum_{i=1}^{n}\lambda_i Tx_i)\| &= \|\frac{1}{k}(T(\sum_{j=1}^{n-1}\mu_j u_j) - \sum_{j=1}^{n-1}\mu_j u_j')\| \\ &\leq \|\frac{1}{k}(T(\sum_{j=1}^{n-1}\mu_j u_j) - \sum_{j=1}^{n-1}\mu_j Tu_j)\| + \sum_{j=1}^{n-1}\frac{1}{k}\mu_j \| Tu_j - u_j' \| \\ &= \mathrm{I} + \mathrm{II},\end{aligned}$$

where $\mathrm{I} = \|\frac{1}{k}(T(\sum_{j=1}^{n-1}\mu_j u_j) - \sum_{j=1}^{n-1}\mu_j Tu_j)\|$ and $\mathrm{II} = \sum_{j=1}^{n-1}\frac{1}{k}\mu_j \| Tu_j - u_j' \|$.

Puttint $t=\sum_{1\leq i\leq j\leq n}\lambda_i\lambda_j(\|x_i-x_j\|-\frac{1}{k}\|Tx_i-Tx_j\|)$, and using the properties of γ and γ_{n-1}, we get

$$\begin{aligned}\gamma(\mathrm{II})&\leq\sum_{j=1}^{n-1}\mu_j\gamma(\frac{1}{k}\|Tu_j-u_j\|)\\&\leq\sum_{j=1}^{n-1}\lambda_j\lambda_n(\|x_j-x_n\|-\frac{1}{k}\|Tx_j-Tx_n\|)\\&\leq t,\end{aligned}$$

hence $\mathrm{II}\leq\gamma^{-1}(t)$, the inverse function of γ. On the other hand, by the assumption of induction we obtain

$$\begin{aligned}\gamma_{n-1}\ (\mathrm{I})&\leq\sum_{1\leq i\leq j\leq n-1}\mu_i\mu_j(\|u_i-u_j\|-\frac{1}{k}\|Tu_i-Tu_j\|)\\&\leq\sum_{1\leq i\leq j\leq n-1}\mu_i\mu_j(\|u_i-u_j\|-\frac{1}{k}\|u_i'-u_j'\|+\frac{1}{k}\|u_i'-Tu_i\|+\frac{1}{k}\|u_j'-Tu_j\|)\\&\leq\sum_{1\leq i\leq j\leq n-1}(\frac{\lambda_i\lambda_j}{1-\lambda_n})\quad(\|x_i-x_j\|-\frac{1}{k}\|Tx_i-Tx_j\|)+2\mathrm{II}.\\&\leq 2t+2\gamma^{-1}(t),\end{aligned}$$

hence $\mathrm{I}\leq\gamma_{n-1}^{-1}\ (2t+2\gamma^{-1}(t))$. Thus

$$\|\frac{1}{k}(T(\sum_{i=1}^{n}\lambda_ix_i)-\sum_{i=1}^{n}\lambda_iTx_i)\|\leq\gamma_{n-1}^{-1}\ (2t+2\gamma^{-1}(t))+\gamma^{-1}(t).$$

We define $\gamma_n^{-1}(t)=\gamma_{n-1}^{-1}(2t+2\gamma^{-1}(t))+\gamma^{-1}(t)$ inductively for $n\geq 3$ with $\gamma_2=\gamma$. Then as easily checked from the properties of functions $\gamma_k\varepsilon\Gamma$, $k<n$, we have $\gamma_n\varepsilon\Gamma$. Hence $\gamma_n(\frac{1}{k}\|T(\sum_{i=1}^{n}\lambda_ix_i)-\sum_{i=1}^{n}\lambda_iTx_i\|)\leq t$. This completes the proof.

Corollary 3.2. Let X be a uniformly convex Banach space, C be a bounded closed nonempty convex subset of X. If $T: C\to X$ is a lipschitzian mapping with lipschitzian constant k, then for all $x_i\varepsilon C$, $\lambda_i\geq 0$, $i=1,2,\ldots,n$ with $\sum_{i=1}^{n}\lambda_i=1$ there exists a $\lambda_n\varepsilon\Gamma$ such that

$$\gamma_n\left(\frac{1}{k}\| T(\sum_{i=1}^{n}\lambda_i x_i)-\sum_{i=1}^{n}\lambda_i Tx_i \|\right)\leq \sum_{1\leq i\leq j\leq n} \lambda_i\lambda_j\left(\| x_i-x_j \| -\frac{1}{k}\| Tx_i-Tx_j \|\right).$$

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