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On algebras which resembles the local Weyl algebra $\hat{D}_n(K)$

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Let K be an algebraically closed field of characteristic zero and let $\hat{Q}_n(K)=k[[x_1,\cdots,x_n]]$ be the formal power series ring over K in n veriables. According to Björk [1], we denote by $\hat{D}_n(K)$ the subring of $\mathrm{End}_K(\hat{Q}_n(K))$ generated over K by the left multiplications by elements of $\hat{Q}n(K)$ and partial differentials $\partial_i=\frac{\partial}{\partial x_i}$

$$\hat{D}_n(K)=\hat{Q}_n(K)\langle\partial_1,\cdots,\partial_n\rangle.$$

where $\partial_i x_j-x_j\partial_i=\delta_{ij}$ (Kronecker's delta) and $\partial_i\partial_j=\partial_j\partial_i$. The ring $\hat{D}_n(K)$ has the Σ-filtration $\{\Sigma_v\}_{v\geq 0}$ such that $\Sigma_0=\hat{Q}_n(K)$ and $\Sigma_v=\{\Sigma_\alpha f_\alpha\partial^\alpha ; f_\alpha\in\hat{Q}_n(K)$ and $\partial^\alpha=\partial_1^{\alpha_1}\cdots\partial_n^{\alpha_n}$ with $|\alpha|=\alpha_1+\cdots+\alpha_n\leq v\}$ and that the associated graded ring $gr_\Gamma(\hat{D}_n(K))$ is a polynomial ring over $\hat{Q}_n(K)$ in n variables. Moreover, $\hat{D}_n(K)$ has weak global dimension n, i.e., w.gl. dim $(\hat{D}_n(K))=n$.

In the present article, we consider whether or not these conditions are sufficient to characterize the ring $\hat{D}_n(K)$.

2. Structure theorems

To simplify the notations, we denote $\hat{Q}_n(K)$ by R. let A be a (not necessarily commutative) ring finitely generated over R. Consider the following three conditions on A :

(i) A has a Σ-filtration $\{\Sigma_v\}_{v\geq 0}$ such that $\Sigma_0=R$, Σ_1 generates A over R, $\Sigma_v\cdot\Sigma_w\subset\Sigma_{v+w}$ for any $v,w\geq 0$ and $A=\bigcup_{v\geq 0}\Sigma_v$;

(ii) The associated graded ring $g_\Sigma^r(A)=\bigoplus_{v\geq 0}\Sigma_v/\Sigma_{v-1}$ is a polynomial

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ring $R[y_1,\cdots,y_m]$ in m variables;

(iii) w.gl.dim(A)=n.

If A satisfies the above conditions (i) and (ii), we call it a *pre-W-algebra* over R. We denote by L the free R-module $\Sigma_1/\Sigma_0=\bigoplus_{i=1}^{m} Ry_i$.

Lemma 2.1. *Let* A *be a pre-w-algebra over* R. *Then we have the following*:

(1) *Let* $Y_1,\cdots,Y_m$ *be elements of* Σ_1 *such that* $y_i\equiv Y_i$ mod Σ_0 *for any* i. *Then* A *is generated by* $Y_1,\cdots,Y_m$ *over* R, *which we write as* $A=R\langle Y_1,\cdots,Y_m\rangle$.

(2) *For any* $y\in L$ *and* $a\in R$, *define* y[a] *by*

$$y[a]=Ya-aY$$

for $Y\in\Sigma_1$ *with* $y\equiv Y(\text{mod }\Sigma_0)$. *Then* y[a] *is independent of the choice of* Y, *and* y *is considered as a* K*-derivation on* R. So, *we have an* R*-linear map* $\rho: L\rightarrow Der_K(R)$; *we write* y[a] *as* $\rho(y)(a)$ as well.

(3) *Define a bracket product* [y,z] *on* L *by*

$$[y,z]\equiv YZ-ZY \quad (\text{mod } \Sigma_0)$$

for $Y,Z\in\Sigma_1$ *with* $y\equiv Y(\text{mod }\Sigma_0)$ *and* $z\equiv Z(\text{mod }\Sigma_0)$. *Then* [y,z] *is well-defined and* ρ *is a Lie-algebra homomorphism,* i.e., $\rho([y,z])=[\rho(y),\rho(z)]$.

Proof. (1) For andy $f\in A$, we define v(f) as the smallest integer r with $f\in\Sigma_r$. If v(f)=r, there exists $F_r(y_1,\cdots,y_m)\in R[y_1,\cdots,y_m]$=the r-th homogeneous part of $g^r_\Sigma(A)$ such that $f-F_r(Y_1,\cdots,Y_m)\in\Sigma_{r-1}$. By induction on v(f), we can verify the assertion straightforwardly.

(2) Replace Y by Y+b with $b\in R$. Then we have

$$(Y+b)a-a(Y+b)=Ya-aY,$$

whence Y[a] is independent of the choice of Y. Furthermore, we have

$$\begin{aligned} y[ab]&=Y(ab)-(ab)Y=(aY+y[a])b-abY\\ &=abY+ay[b]+y[a]b-abY=ay[b]+y[a]b. \end{aligned}$$

So, y[ ] is a K-derivation on R.

(3) The assertion can be verified by a straightforward computation.
Q.E.D.

The structure of a pre-W-algebra over R is given in the following :

Theorem 2.2. (1) *Let* A *be a pre-w-algebra over* R. *Let* $Y_1,\cdots,Y_m$ *be elements of* Σ_1 *as chosen in the previous lemma. Write*

(2.0) $Y_iY_j - Y_jY_i = \sum_{k=1}^{m} \rho_{ij,k} Y_k + \sigma_{ij}, \quad 1 \leq i, j \leq m,$

were $\rho_{ij,k}, \sigma_{ij} \in R$. *Then we have the following equalities :*

(2.1) $\sum_{\ell=1}^{m} (\rho_{ij,\ell}\rho_{\ell k,s} + \rho_{jk,\ell}\rho_{\ell i,s} + \rho_{ki,\ell}\rho_{\ell j,s})$
$= y_i[\rho_{jk,s}] + y_j[\rho_{ki,s}] + y_k[\rho_{ij,s}], \quad 1 \leq i, j, k, s \leq m.$

(2.2) $\sum_{\ell=1}^{m} (\rho_{ij,\ell}\, \sigma_{\ell k} + \rho_{jk,\ell}\, \sigma_{\ell i} + \rho_{ki,\ell}\sigma_{\ell j})$
$= y_i[\sigma_{jk}] + y_j[\sigma_{ki}] + y_k[\sigma_{ij}], \quad 1 \leq i, j, k \leq m.$

(2.3) $\rho_{ij,k} = -\rho_{ji,k}, \quad \sigma_{ij} = -\sigma_{ji}, \leq i, j, k \leq m.$

The elements $\{\rho_{ij,k} ; 1 \leq i, j, k \leq m\}$ *are determined uniquely by the Lie algebra L and the choice of* R*-free basis* $\{y_1,\cdots,y_m\}$ *of* L.

(2) *Suppose we are given as in Lemma 2.1. the Lie algebra* L *and an* R*-Linear map* $\rho : L \to Der_K R$ *which is a Lie-algebra hmomorphism. For an* R*-free basis* $\{y_1,\cdots,y_m\}$ *of* L, *suppose we are given elements* $\{\sigma_{ij} ; 1 \leq i, j \leq m\}$ *satisfying the conditions* (2.2) *and* (2.3) *above. Then there exists an* R*-algebra* A *with a* Σ*-filtration* $\{\Sigma_v\}_{v \geq 0}$ *such that*

(i) A *is generated over* R *by elements* $Y_1,\cdots,Y_m$;

(ii) *The equalities* (2.0)~(2.3) *hold* ;

(iii) $\Sigma_v = \{\Sigma_\alpha f_\alpha Y^\alpha ; f_\alpha \in R, \ Y_\alpha = Y_1^{\alpha_1} Y_m^{\alpha_m}, \ |\alpha| \leq v\}$ *for any* $v \geq 0$;

(iv) $gr_\Sigma(A) \simeq R[y_1,\cdots,y_m] :=$ *the symmetric algebra of* L *over* R.

Proof. (1) By the definition of $[y_i, y_j]$ in Lemma 2.1, $\{\rho_{ij,k} ; 1 \leq i, j, k \leq m\}$ are the multiplication constants of the Lie algebra L. Hence they are uniquely determined by the choice of the R-free basis $\{y_1,\cdots y_m\}$ of L. if one chooses $\{y_1,\cdots,Y_m\}$ as in Lemma 2.1, then $\{1, Y_1,\cdots Y_m\}$ is an R-free basis of Σ_1. Then the equalities (2.1) and (2.2) follow from the Jacobi identity:

$$[[Y_i,Y_j],Y_k]+[[Y_j,Y_k],Y_i]+[[Y_k,Y_i],Y_j]=0,$$

where $[Y_i,Y_j]=Y_iY_j-Y_jY_i$.

(2) Let $\{Y_1,\cdots,Y_m\}$ be indeterminates and let A be the free R-algebra generated by $Y_1,\cdots Y_m$ modulo the two-sided ideal I generated by

$$\{Y_iY_j-Y_jY_i-\sum_{k=1}^{m}r_{ijk}Y_k-\sigma_{ij}\ ;\ 1\leq i,j,k\leq m\}\ \text{and}$$

$$\{Y_if-fY_i-\rho(y_i)(f)\ ;\ 1\leq i\leq m,\ \ \forall f\in R\}.$$

We write $y_i[f]=\rho(y_i)(f)$ by identifying Y_i's with y_i's in L. We can employ the proof of the Poincare-Birkoff-Witt theorem (cf. Jacobson [2]) without major changes in the present situation to show that every element of A is written uniquely as a linear combination of standard monomials in $Y_1,\cdots,Y_m$ with coefficients in R. In particular, the equalities (2.1) and (2.2) imply that Σ_1 (with the notation in (iii)) is a free R-module generated by $1,Y_1,\cdots,Y_m$. Note that there is a surjective homomorphism $\theta: R[Y_1,\cdots,Y_m]\rightarrow gr_\Sigma(A)$. Its kernel is generated by the relations $y_iy_j-y_jy_i$ and y_if-fy_i, $1\leq i,j\leq m$. But these elements are already zero in $R[Y_1,\cdots Y_m]$. Hence $g_\Sigma^r(A)\simeq R[y_1,\cdots,y_m]$. Q.E.D.

Let A be a pre-W-algebra over R. We are interested in the existence of an R-algebra homomorphism from A to the local Weyl algebra $\hat{D}_n(K)$.

Theorem 2.3. *Let* A *be a pre-W-algebra over* R. *Then the following conditions on* A *are equivalent*:

(1) *There is an* R-*algebra homomorphism* $\bar{\pi}: A\rightarrow\hat{D}_n(K)$ *such that* $\tilde{\rho}(\Sigma_v)$ $\ulcorner\Sigma_v$ *for all* $v\geq 0$ *and* $\tilde{\rho}|\Sigma$ *induces the Lie-algebra homomorphism* $\rho: L:=\Sigma_1/\Sigma_0\rightarrow Der_k(R)$.

(2) *There exists a lifting* $\{Y_1,\cdots,Y_m\}$ *of the* R-*free basis* $\{y_1,\cdots,y_m\}$ *in* Σ_1 *for which* $\sigma_{ij}=0, 1\leq i,j\leq m$.

(3) *There exist* $\{a_i\}_{1\leq i\leq m}$ *in* R *such that*

$$(2.4)\quad \sigma_{ij}=\sum_{\ell=1}^{m}\rho_{ij\ell}\,a_\ell+y_i[a_j]-y_j[a_i],\ \ 1\leq i,\ j\leq m.$$

(4) *There exists an* R*-free submodule* $\tilde{L}$ *of* Σ_1 *such that* $\tilde{L}$ *is closed under the bracket product* $[Y,Z]=YZ-ZY$ *and the natural residue homomorphism* $\pi\mid\tilde{L}\ \tilde{L}\to L$.

Proof (1)→(1). Note that $\hat{D}_n(K)$ acts on R in the natural fashion. So, A acts on R via the homomorphism $\tilde{\rho}$. For $Y\in\Sigma_1$, let $a=\tilde{\rho}(Y)\cdot 1$ and let $Y'=Y-a$. Then, since $\tilde{\rho}(Y)\in\Sigma_1:=\bigoplus_{i=1}^{n}R\frac{\partial}{\partial x_i}+R$, we know that $\tilde{\rho}(Y')\in Der_K(R)$. In particular, $\tilde{\rho}(Y')\cdot 1=0$. Now, for the given lifting $\{Y_1,\cdots,Y_m\}$, we set $Y_i=Y_i-\tilde{\pi}(Y_i)\cdot 1$, $1\leq i\leq m$. Then $\{Y_1,\cdots,Y_m\}$ is a lifting of $\{Y_1,\cdots,y_m\}$ in Σ_1. We assume from the beginning that $Y_i=Y_i$, $1\leq i\leq m$. Then the equality (2.0) implies $\sigma_{ij}=0$ $(17\leq i,j\leq m)$ because $\tilde{\rho}(Y_i)\in Der_K(R)$.

(2)→(3). Suppose $\{Y_1,\cdots,Y_m\}$ is the given lifting of $\{y_1,\cdots y_m\}$ and $\{Y_1,\cdots,Y_m\}$ is a lifting for which $\sigma_{ij}=0$ when we write

$$(2.0)\quad Y_iY_j-Y_jY_i=\sum_{k=1}^{m}r_{ijk}Y_k+\sigma_{ij},\quad 1\leq i,j\leq m.$$

Then $Y_i=Y_i+a_i$ with $a_i\in R$. Replacing Y_i in (2.0)′ by this expression, we obtain the equality (2.4).

(3)→(2). Conversely, if we are given $\{a_i\}_{1\leq i\leq m}$ satisfying (2.4), set $Y_i=Y_i+a_i$. Then $\{Y_1,\cdots,Y_m\}$ is a lifting of $\{y_1,\cdots,y_m\}$ for which $\sigma_{ij}=0$.

(2)→(4). Let $\{Y_1,\cdots,Y_m\}$ be as in (2) above. Let $\tilde{L}$ be the R-submodule of Σ_1 generated by $Y_1,\cdots,Y_m$. Then $\tilde{L}$ is a free R-module. Since $\sigma_{ij=0}$, we readily verify that $[Y,Z]\in\tilde{L}$ for any $Y,Z\in\tilde{L}$ Clearly, π induces an isomorphism between $\tilde{L}$ and L.

(4)→(1). Define $\tilde{\rho}\ \tilde{L}\to Der_K(R)$ by $\tilde{\rho}(Y)=\rho(\pi(y)$, Extend this to Σ_1 in a natural fashion by putting $\tilde{\rho}|_{\Sigma_0}=id_R$. Furthermore, we extend $\tilde{\rho}$ to the free R-algebra F generated by $Y_1,\cdots,Y_m$ as follows. For an element $Y_{i_1}f_{j_1}\cdots Y_{i_r}f_{j_r}$ of F with $Y_{i_j}\in\{Y_1,\cdots,Y_m\}$ and $f_{i_j}\in R$, define

$$Y_{i_1}f_{j_1}\cdots Y_{i_r}f_{j_r}(a)=y_{i_1}[f_{j_1}[y_{i_2}[f_{j_2}[\cdots[f_{j_r}(a)]\cdots]]]],$$

where $y_{i_j}=\pi(Y_{i_j})$ and $f[b]:=fb\in R$. In view of (2) of Theorem 2.2, A is identified with the residue ring of F by the two-sided ideal I considered in Theorem 2.2. So, in order to have $\tilde{\rho}$ as above, we have only to show that

$$Y_i[y_j[a]]-y_j[y_i[a]]=\sum_{k=1}^{m}\rho_{ij,k}y_k[a] \quad \text{and}$$

$$y_i[fa]=fy_i[a]+y_i[f]a$$

for $a\in R$. These equations hold, in fact, because $\rho: L\to Der_K(R)$ being a Lie-algebra homomorphism implies

$$y_i[y_j[a]]-y_j[y_i[a]]=[y_i, y_j][a]=\sum_{k=1}^{m}\rho_{ij,k}\, y_k[a]$$

and the second equality above. Q.E.D.

If a pre-W-algebra A over R satisfies one of the equivalent conditions in Theorem 2.3, we call A a *W-algebra* over R.

Remark 2.4 (1) Suppose that $\rho: L\to Der_K(R)$ is an isomorphism. Then, as an R-free basis $\{y_1,\cdots,y_m\}$ of L, we can take $y_i=\rho^{-1}(\frac{\partial}{\partial x_i})$. Then $\rho_{ij,k}=0$ for all $1\leq i,j,k\leq m$. So the case with all $\rho_{ij,k}=0$ can take place. We then say that L is *essentially abelian*.

(2) Suppose L is essentially abelian. Let $\{y_1,\cdots,y_m\}$ be an R-free basis of L such that $[y_i, y_j]=0, 1\leq i,j\leq m$ and let $\{y_i,\cdots Y_m\}$ be such that $y_i\equiv Y_i(\text{mod } \Sigma_0)$ and $Y_iY_j-Y_jY_i=c_{ij}\in K^*=K-(0)$ for $1\leq i, j\leq m$ $i\neq j$. Suppose that $\rho(y_i)(M)\in \mathfrak{m}$, where $\in$ is the maximal ideal of R. Then we cannot find $\{a_i\}_{1\leq i\leq m}$ so that the equallity (2.4) holds. there exists an R-algebra A satisfying these conditions. In fact, we can take A to be the residue ring of an R-free algebra F generated by $Y_1,\cdots,Y_m$ modulo the two-sided ideal I as considered in Theorem 2.2,(2). Then ρ cannot be extended to an R-algebra homomorphism $\tilde{\rho}: A\to \hat{D}_n(K)$ as considered in Theorem 2.3.

3. Case L is essentially abelian

we begin with the following:

Lemma 3.1. *Let* A *be a W-algebra over* R *with an* R*-algebra homomorphism* $\tilde{\rho}$: $A \to \hat{D}^n(K)$ *which is an extension of the Lie-algebra homomorphism* $\rho: L \to Der_K(R)$. *Then we have* w.gl.dim(A)$\geq$n.

Proof. Note that any element ξ of A can be expressed as $\xi = \Sigma_\alpha f_\alpha y^\alpha$, where $f_\alpha \in R$ and $Y_\alpha = Y_1^{\alpha_1} \cdots Y_m^{\alpha_m}$ (cf. the equality $Ya - aY = y[a]$ in Lemma 2.1). Furthermore, this expression is unique. Indeed, if we have a nontrivial expression $\Sigma_\alpha f_\alpha Y^\alpha = 0$ then this yields a homogeneous nontrivial relation

$$\sum_{|\alpha|=v} f_\alpha y^\alpha = 0, \quad y^\alpha = y_1^{\alpha_1} \cdots y_m^{\alpha_m},$$

where $v = \max\{|\alpha| ; f_\alpha \neq 0\}$. This contradicts the hypothesis that $gr_\Sigma(A)$ is a polynomial ring in $y_1, \cdots, y_m$ over R. Hence A is a free R-module, whence A is R-flat as a left R-module. Similarly, ξ can be expressed uniquely as $\xi = \Sigma_\beta y^\beta g_\beta$ So, a is R-flat as a right R-module. Hence A is R-flat as a ring. In view of Björk [1; Cor. 2.9, p.42], we have

$$(*) \qquad \text{w.dim}_R (A \underset{R}{\otimes} M) \leq \text{w.dim}_A (A \underset{R}{\otimes} M)$$

for any left R-module M. Take an R-module $K = R/\mathfrak{.}$ with $\mathfrak{.} = (x_1, \cdots, x_n)R$. Then, by the theory of syzyzy, we know that $\text{w.dim}_R(K) = n$; in fact, $Tor_n^R(K,K) = K \neq (0)$. Then the above inequality (*) implies that $\text{w.dim}_A (A \underset{R}{\otimes} K) \geq n$. Hence
w.gl.dim(A)$\geq$n. Q.E.D.

We shall be concerned with the condition w.gl.dim(A)$=$n for a W-algebra over R.

Theorem 3.2. *Let* A *be a W-algebra over* R *with an* R*-algebra* *homomorphism* $\tilde{\rho}: A \to \hat{D}_n(K)$. *Suppose that* L *is essentially abelian and* A *has* w.gl.dim(A)=n. *Then* $\tilde{\rho}$ is an injection.

Proof. Let $\tilde{\rho}_1 = \tilde{\rho}|L$, where $\tilde{L}$ is an R-free submodule of Σ_1 isomorphic to L as a Lie algebra (cf. Theorem 2.3). Then there exists an R-free basis $\{Y_1, \cdots, Y_m\}$ of $\tilde{L}$ such that $Y_i Y_j = j Y_j Y_i$ for $1 \leq i, j \leq m$. Let $\tilde{L} = \bigoplus_{i=1}^{m} KY_i$ and let $Q = \mathrm{Ker}(\tilde{\rho}_1 | L_0)$. Then $\tilde{L}_0 \simeq Q \subset \tilde{\rho}_1(\tilde{L})$ is a direct sum as Lie algebras and Q is contained in the center of A. Let B be the R-subalgebra generated of $\hat{D}_n(K)$ by $\tilde{\rho}_1(\tilde{L})$ and let J be the two-sided ideal of A generated by Q. Then $B \simeq A/J$ and B is a W-algebra over R. Indeed, we may take $\{Y_1, \cdots, Y_m\}$ so that $\{Y_{r+1}, \cdots Y_m\}$ is a K-basis of Q. Let $Y_i = \tilde{\rho}_1(Y_i)$, $1 \leq i \leq r$. Then B is generated by $Y_r, \cdots, Y_r$ over R which act on R via the derivations $\delta_i = y_i[\quad]$, $1 \leq i \leq r$ Note that $\{Y_1, \cdots, Y_r\}$ are linearly independent over R. We claim:

Lemma 3.3. $\{\delta_1, \cdots \delta_r\}$ *are algebraically independent over* R. *Namely, if* $\sum_{\gamma} f_\gamma \delta^\gamma = 0$ *with* $f_\gamma \in R$ *and* $\delta^\gamma = \delta_1^{\gamma_1} \cdots \delta_r^{\gamma_r}$ *then* $f_\gamma = 0$ *for all* γ.

Proof. Denote by Q(R) the quotient field of R. We can find $\triangle_2, \cdots \triangle_r \in \bigoplus_{i=1}^{r} R\delta_i$ satisfying the following conditions:

(1) $\bigoplus_{i=1}^{r} Q(R)\delta_i = \bigoplus_{i=1}^{r} Q(A)\triangle_i$;

(2) if we express $\triangle_i = \sum_{j=1}^{n} a_{ij} \partial_j$ with $a_{ij} \in R$ and $\partial_j = \frac{\partial}{\partial x_j}$ and define s_i $\min\{j ; a_{ij} \neq 0\}$ then $s_1 < s_2 < \cdots < s_r$.

Suppose we have a nontrivial relation $\sum f_\gamma \delta^\gamma = 0$. Let $v = \max\{|\gamma| ; f_\gamma \neq 0\}$. Expressing δ_i as a Q(R)-linear combinations of $\triangle_j$'s and substituting it for δ_i in $\sum f_\gamma \delta^\gamma = 0$, we obtain a nontrivial relation $\sum_\gamma g_\gamma \triangle^\gamma = 0$ with $\max\{|\gamma| ; g_\gamma \neq 0\} = v$. Expressing then $\triangle_\gamma$ in terms of $\partial^\beta = \partial_1^{\beta_1} \cdots \partial_n^{\beta_n}$, we obtain

$$(*) \quad \sum_{|\gamma|=v} (g_\gamma \prod_{i=1}^{r} (a_{is_i})^{\gamma_i}) \partial^{\tilde{\gamma}} + \cdots = 0,$$

where $\tilde{\gamma} = (\gamma_1, \cdots, \gamma_r, 0 \cdots 0)$ if $\gamma = (\gamma_1, \cdots, \gamma_r)$. Among g_γ's with $|\gamma| = v$

and $g_\gamma \neq 0$, let $(\alpha_1,\cdots,\alpha_r)$ be the largest with respect to the lexicographic relation : $(\gamma_1\cdots\gamma_r) \geq (\gamma_1',\cdots,\gamma_r')$ if and only if $\gamma_1=\gamma_1',\cdots,\gamma_{t-1}=\gamma_{t-1}', \gamma_t \geq \gamma_t'$. Then $(g_\alpha \prod_{i=1}^{r}(a_{is_i})^{\alpha_i})\partial^{\tilde{\alpha}}$ has no other terms in $(*)$ to cancel with. This is a contradiction. Q.E.D.

Proof of Theorem 3.2 resumed. The above lemma implies that B is isomorphic to a W-algebra over R generated by $Y_1,\cdots,Y_r$. Since any element ξ of A is expressed uniquely in the form

$$(**) \qquad \xi=\sum_{\gamma} f_\gamma Y^\gamma+\eta, \quad f_\gamma \in R \text{ and } \eta \in J,$$

where $Y^\gamma = Y_1^{\gamma_1}\cdots Y_r^{\gamma_r}$, we know that A/J is isomorphic to B.

Now we can easily show that $A \simeq B[Y_{r+1},\cdots,Y_m]$, a polynomial ring in $Y_{r+1},\cdots,Y_m$ over B (cf. the above expression $(**)$ of ξ). By Björk [1 ; Th. 3.4, p.43], we have $\text{w.gl.dim}(A)=\text{w.gl.dim}(B)+(m-r) \geq n+m-r$ (cf. Lemma 3.1). By the hypothesis $\text{w.gl.dim}(A)=n$, we have $m=r$. This implies $J=(0)$. Hence $A \simeq B$. Q.E.D.

A W-algebra A over R is called a *W-subalgebra of* $\hat{D}_n(K)$ provided $\tilde{\rho}$ is injective.

Theorem 3.4. *There is a one-to-one correspondemce between the set of W-subalgebras of $\hat{D}^n(K)$ and the set of R-submodules L of* $\text{der}_K(R)$ *satisfying the conditions :*

(L-1) $\mathfrak{L}$ *is a free R-submodule of* $\text{Der}_K(R)$;
(L-2) $\mathfrak{L}$ *is closed under the bracket product of* $\text{Der}_K(R)$.

Proof. Let A be a W-subalgebra of $\hat{D}_n(K)$. Then we can find an R-free submodule $\tilde{L}$ of Σ_1 which is isomorphic to $L := \Sigma_1/\Sigma_0$. Since $\tilde{\rho}$ is injective, so is $\rho : L \to \text{Der}_K(R)$. Hence $\mathfrak{L}$ is an R-free submodule of $\text{Der}_K(R)$. Since $\rho \cdot (\pi \mid \tilde{L})$ is a Lie-algebra homomorphism, L is closed under the bracket product (cf. theorem 2.3). Conversely, let $\mathfrak{L}$ be an R-submodule of $\text{Der}_K(R)$ satisfying the condition (L-1) and (L-2). Let $\{Y_1,\cdots,Y_m\}$ be an R-free basis of $\mathfrak{L}$. Then we have :

(1) $Y_iY_j - Y_jY_i = \sum_{k=1}^{m} r_{ijk}Y_k$, $1 \leq i, j \leq m$,

(2) $Y_if - fY_i = y_i[f]$ for $f \in R$ and $1 \leq i \leq m$.

Construct an R-algebra A as in Theorem 2.2, (2). Then the natural R-algebra homomorphism $A \to \hat{D}_n(K)$ is injective (cf. the proof of Lemma 3.3). Q.E.D.

A W-subalgebra A of $\hat{D}_n(K)$ is said to be *of maximal rank* if rank $\tilde{L} = n$. We shall consider the case $n = 1$. then L is essentially abelian. Hence there exists an R-algebra homomorphism $\bar{\rho}: A \to \hat{D}_1(K)$ which must be injective by virtue of Theorem 3.2. We set $Y = Y_1$, a free generator of the R-module $\tilde{L}$(cf. Theorem 2.3). Then we have $Yx - xY = f$, where $f = x^r u$ with $u \in R^*$. Replacing Y by $u^{-1}Y$, we may assume that $f = x^r$. We shall show:

Lemma 3.5. $\mathrm{Tor}_2^A(K,K) = K$ *if* $r \geq 2$, *while it is zero if* $r = 1$. $\mathrm{Tor}_1^A(K,K) = K$ *if* $r = 1$.

Proof. Suppose $r > 0$. Then K is a two-sided A-module. As a right A-module, K has the following free A-module resolution:

$$0 \longrightarrow e_2A \xrightarrow{\bar{\varphi}_1} e_1A \oplus e_1'A \xrightarrow{\varphi_0} e_0A \xrightarrow{\varepsilon} K \longrightarrow 0,$$

where ε is the natural residue homomorphism and φ_i $(i = 0,1)$ is given as:

$$\varphi_0(e_1) = e_0Y,\ \varphi_0(e_1') = e_0x \ \text{and}\ \varphi_1(e_2) = e_1x - e_1'(Y + x^{r-1}).$$

Take the tensor product of this sequence with a left A-module $K = Av$ to obtain the complex:

$$0 \to e_2A \otimes_A Av \xrightarrow{\bar{\varphi}_1} (e_1A \otimes_A Av) \oplus (e_1'A \otimes Av) \xrightarrow{\bar{\varphi}_0} e_0A \otimes_A Av \to 0,$$

where we can make the identification: $e_iA \otimes_A Av = e_i \otimes Kv$ for $e_i = e_0, e_1, e_1'$

and e_2. Then it is clear that $\bar{\varphi}_1=\bar{\varphi}_0=0$ if $r\geq 2$. Hence $Tor_2^A(K,K)=K$ if $r\geq 2$. If $r=1$, then $\bar{\varphi}_1(e_2'\otimes v)=-e_1\otimes v$, whence $\bar{\varphi}_1$ is injective. So, $Tor_2^A(K,K)=0$ if $r=1$. If $r=1$, $Tor_1^A(K,K)=K$ because $\bar{\varphi}_0=0$.
Q.E.D.

If $n\geq 2$, we know little on W-subalgebras of $\hat{D}_n(K)$ even if it is of maximal rank. We shall give two partial results.

Proposition 3.6. *Let A be a W-subalgebra of maximal rank of* $\hat{D}_n(K)$ *corresponding to a Lie subalgebra* $\mathfrak{L}=\bigotimes_{i=1}^{n} RY_i$ *with* $Y_i=x_i^{r_i}\frac{\partial}{\partial x_i}$ *and* $r_i\geq 1$. *Then we have*

$\mu:=\max\{v\,;\,Tor_v^A(K,K)\neq 0\}=2\#\{i\,;\,r_i\geq 2\}+\#\{i\,;\,r_i=1\}$. *Hence* $r_i=1$ *for all* i *provided* w.gl.dim(A)=n.

Proof. Let S_i be the free algebra generated by Y_i over a one-dimensional polynomial ring $K[x_i]$ modulo the two-sided ideal generated by $Y_ix_i-x_iY_i=x_i^{r_i}$. Sice $Y_iY_j=Y_jY_i$ and $x_iY_j=Y_jx_i$ if $i\neq j$, A is isomorphic to

$$(S_1\otimes_K S_2\otimes_K\cdots\otimes_K S_n)_{K[x_1,\ldots,x_n]}\otimes R,$$

where $S_1\otimes_K\cdots\otimes_K S_n$ is regarded as an algebra over $K[x_1,\ldots,x_n]$. Consider a complex

$$(\tilde{C}_1):0\to e_2S_1\xrightarrow{\varphi_1}e_1S_1\oplus e_1S_1\xrightarrow{\varphi_0}e_0S_1\xrightarrow{\varepsilon}K\to 0,$$

which is defined in the same fashion as in the proof of Lemma 3.5 with A replaced by S_1. It is a resolution of the two-sided S_1-module K by free fight S_1-modules. The complex $\tilde{C}:=(\tilde{C}_1\otimes_K\cdots\otimes_K\tilde{C}_n)_{K[x_1,\ldots,x_n]}\otimes R$ i a resolution of the two-sided A-module K by free right A-modules. Let C_i (resp. C) be the complex obtained from $\tilde{C}_i$ (resp. $\tilde{C}$) by replacing K by 0. Then, taking the tensor products with the left A-module K, we obtain $\bar{C}:=C\otimes_A K\simeq\bar{C}_1\otimes_k\cdots\otimes_K\bar{C}_n$, where $\bar{C}_i=C_i\otimes_A K$. By the Kunneth formula for homolgies, we have

$$\mathrm{Tor}_v^A(K,K) \simeq \bigoplus_{v_1+\cdots+v_n=v} \mathrm{Tor}_{v_1}^{S_1}(K,K) \otimes_K \cdots \otimes_K \mathrm{Tor}_{v_n}^{S_n}(K,K)$$

Thence we obtain the stated formula in view of Lemma 3.5.

Q.E.D.

Proposition 3.7. *Let* A *be a W-subalgebra of maximal rank of* $\hat{D}_2(K)$ *corresponding to a Lie subalgebra* $L = RY_1 + RY_2$ *with* $Y_i g \frac{\partial}{\partial x_i}$, *where* $h = x_1 f + x_2 g \in M := Rx_1 + Rx_2$. *Suppose that* h *is a homogeneous polynomial in* x_1 *and* x_2. *Then* $\mathrm{Tor}_3^A(K,K) \neq 0$ *and* $\mathrm{Tor}_4^A(K,K) = 0$.

Proof. We have the following relations:

$$Y_1Y_2 - Y_2Y_1 = -h_{x_2}Y_1 + h_{x_1}Y_2$$
$$Y_1x_1 - x_1Y_1 = h = Y_2x_2 - x_2Y_2$$
$$Y_1x_2 - x_2Y_1 = 0 = Y_2x_1 - x_1Y_2$$

where $h_{x_i} = \frac{\partial h}{\partial x_i}$. Construct a complex of right A-modules:

$$0 \to e_3A \xrightarrow{\varphi_2} e_2A \oplus e_2'A \oplus e_2^*A \oplus e_2^{**}A \xrightarrow{\varphi_1}$$
$$e_1A \oplus e_1'A \oplus e_1^*A \oplus e_1^{**}A \xrightarrow{\varphi_0} e_0A \xrightarrow{\varepsilon} K \to 0$$

where:

(0) K is the two-sided A-module with $x_i \cdot 1 = y_i \cdot 1 = 0$ for $i = 1, 2$;

(i) $\varepsilon(e_0) = 1$;

(ii) $\varphi_0(e_1) = e_0Y_1$, $\varphi_0(e_1') = e_0x_1$, $\varphi_0(e_1^*) = e_0Y_2$, $\varphi_0(e_1^{**}) = e_0x_2$;

(iii) $\varphi_1(e_2) = e_1x_1 - e_1'(Y_1 + f) - e_1^{**}g$, $\varphi_1(e_2') = -e_1'f + e_1^*x_2$
$- e_1^{**}(Y_2 + g)$, $\varphi_1(e_2^*) = e_1x_2 - e_1^{**}Y_1$, $\varphi_1(e_2^{**}) = -e_1'Y_2 + e_1^{**}x_1$;

(iv) $\varphi_2(e_3) = e_2x_2(Y_1 + g + h_{x_2}) + e_2'x_1(Y_1 + f + h_{x_1})$.
$- e_2^*x_1(Y_2 + g + h_{x_2}) - e_2^{**}x_2(Y_1 + f + h_{x_1})$

It is straightforward to show that this comploex is a resolution of K right free A-modules. The stated result follows from this observation.

Q.E.D.

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