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TOTAL ABSOLUTE CURVATURE OF ORDER k OF IMMERSED SURFACES IN E^m

Yong-Soo Pyo

1. Introduction.

S.S.Chern and R.K.Lashof ([4]) studied the total absolute curvature of immersed manifolds in a higher Euclidean space firstly through ~~the Lipschitz-killing curvature, and N.H.Kuiper ([5]) who studied this~~ area was contemporary with them.

Later, many mathematicians studied for the total absolute curvature (or total mean curvature) of immersed manifolds (B.Y.Chen, T.J.Willmore, K.Yano and Y.T.Shin etc.).

Let M be an immersed surface in a Euclidean m-space E^m. For a unit normal vector e at p in M, the i-th mean curvatures $K_i(p,e)$ ($i=1,2$) of the immersion at (p,e) are defined by

$$K_1(p,e)=\frac{1}{2}[k_1(p,e)+k_2(p,e)],$$

and

$$K_2(p,e)=k_1(p,e)\cdot k_2(p,e),$$

where $k_i(p,e)$ ($i=1,2$) are the eigenvalues (i.e, principal curvatures) of the second fundamental form at (p,e) (see [1,I]). And the total absolute curvatures $TA_i(k)$ of order k of M are defined by

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$$TA_i(\mathrm{k})=\int_{B_v} \mid K_i(p,e) \mid ^k d\sigma \wedge dv, \quad i=1,2,$$

where B_v is the unit normal bundle of M and $d\sigma \wedge dv$ is the volume element of B_v. We can easily know that $TA_2(1)$ is the total absolute curvature of M.

In this paper, we shall study some properties of the total absolute curvatures $TA_1(\mathrm{k})$ and $TA_2(\mathrm{k})$ of order k for some special immersed surfaces E^m.

2. Flat surfaces in E^m

Let M be an immersed surface in a Euclidean space E^m of dimension m. We choose a local field of orthonormal frames $e_1,e_2,\xi_3,\cdots,\xi_m$ in E^m such that, restricted to M, the vectore e_1,e_2 are tangent to M (and,consequently, the remaining vectors $\xi_3,\cdots,\xi_m$ are normal to M).

In this paper, we shall make use the following convention on the ranges of indices ;

$$1\leqq i,j,k,l,\cdots\leqq 2, \qquad 3\leqq r,s,\cdots\leqq m$$

unless otherwise ststed.

In terms of canonical forms ω_i and the connection forms ω_{ij}, the structure equations on the surface M are given as follows ;

$$d\omega_i=\Sigma\omega_{ij}\wedge\omega_j, \quad \omega_{ij}+\omega_{ji}=0,$$

$$d\omega_{ij}=\Sigma\omega_{ik}\wedge\omega_{kj}+\Omega_{ij}, \quad \Omega_{ij}=\frac{1}{2}\Sigma R_{ijkl}\ \omega_k\wedge\omega_l,$$

where Ω_{ij}(resp. R_{ijkl}) denotes the curvature form (resp. curvature tensor) on the surface M. Since $\omega_r=0$, by Cartan's lemma, we may write

$$\omega_{ir}=\Sigma h^r_{ij}\omega_j, \quad h^r_{ij}=h^r_{ji}.$$

Then the Gauss curvature G and the mean curvature α are given respectively by

$$G=\Sigma(h^r_{11}h^r_{22}-h^r_{12}h^r_{12}),$$
$$\alpha=\frac{1}{2}[\Sigma(h^r_{11}+h^r_{22})^2]^{\frac{1}{2}}.$$

Furthermore, we can choose normal vectors $e_3,e_4,\cdots,e_m$ at p in M satisfying the following two equations ([6], [10]);

$$K_1(p,e)=\frac{1}{2}\sum_{r=3}^{5}(h^r_{11}+h^r_{22})\ \cos\theta_r,$$

$$K_2(p,e)=\sum_{r=3}^{5}\lambda_{r-2}(p)\ \cos^2\theta_r,\ \ \lambda_1\geqq\lambda_2\geqq\lambda_3$$

for any unit normal vector $e=\Sigma\cos\theta_r e_r$ at p, where $\lambda_{r-2}(p)$ is the determinant of (h^r_{ij}).

Remark. In [6], Pyo has proved that $\lambda_3\leqq 0$, and a compact surface M is homeomorphic to a 2-sphere if $\lambda_3=0$.

In this chapter, we shall take a local field of orthonormal frames $e_1,e_2,\cdots,e_m$ satisfying the above equations.

Let M be a compact flat surface in E^m, *i.e*, $G=\lambda_1+\lambda_2+\lambda_3=0$. For a unit normal vector $e=\Sigma\cos\theta_r e_r$ at p in M, the second mean curvature $K_2(p,e)$ is given by

$$\begin{aligned}K_2(p,e)&=\lambda_1(p)\cos^2\theta_3+\lambda_2(p)\cos^2\theta_4+\lambda_3(p)\cos^2\theta_5\\&=\lambda_1(p)(\cos^2\theta_3-\cos^2\theta_4)+\lambda_3(p)(\cos^2\theta_5-\cos^2\theta_4).\end{aligned}$$

Therefore, for a positive even integer k, we have

$$\int_{S^{m-3}} |K_2(p,e)|^k d\sigma$$

$$=\sum \frac{k!}{r!\,s!}\lambda_1(p)^r\lambda_3(p)^s\int_{S^{m-3}}\sum\frac{(-1)^{u+v}r!\,s!}{t!\,u!\,v!\,w!}\ \cos^{2t}\theta_3\cos^{2u+2v}\theta_4\cos^{2w}\theta_5 d\sigma$$

for nonnegative integers r,s,t,u,v and w such that $r+s=k$, $t+u=r$ and $v+w=s$. If we put

$$T_{t,u+v,w}=\ \sum(-1)^{i+l}\frac{w!}{i!\,l!}\frac{C_{2(t+u+v+i+l+1)}}{C_{2(t+u+v+i+l)+1}}\ B(\frac{2t+2i+1}{2},\frac{2u+2v+2l+1}{2}),$$

where i,l are nonnegative integers such that $i+l\leq w$, C_n is the volume of unit n-sphere $\int_{S^n}$ and B is the Beta function (see [8],[10]), then we have

$$TA_2(k)=\int_M[\int_{S^{m-3}}|K_2(p,e)|^k d\sigma]dv$$

$$=\frac{2C_{m+2k-3}}{C_{2k+3}}\sum\frac{k!}{r!\,s!}\ \sum\frac{(-1)^{u+v}r!\,s!}{t!\,u!\,v!\,w!}\ T_{t,u+v,w}\ \int_M\lambda_1(p)^r\lambda_3(p)^s dv.$$

Hence we have the following theorem.

Theorem 2.1. Let M be a compact flat surface in E^m and k a positive even interger. Then the second total absolute curvature

$$TA_2(k)=\frac{2C_{m+2k-3}}{C_{2k+3}}\ \sum\frac{(-1)^{u+v}\ k!}{t!\,u!\,v!\,w!}\quad T_{t,u+v,w}\ \int_M\lambda_1(p)^{t+u}\lambda_3(p)^{v+w}dv$$

for nonnegative integers t,u,v and w such that $t+u+v+w=k$.

And also, we can prove the following theorem because $\lambda_3=-\lambda_1-\lambda_2$ for a flat surface.

Theorem 2.2. Let M be a compact flat surface in E^m and let k be a positive even integer. Then we have

$$TA_2(k)=\frac{2C_{m+2k-3}}{C_{2k+3}}\sum\frac{(-1)^{s+u}\,k!}{r!\,s!\,t!\,u!}T_{r,t,s+u}\int_M \lambda_1(p)^{r+s}\lambda_2(p)^{t+u}dv$$

for nonnegative integers r,s,t and u such that $r+s+t+u=k$.

Corollary 2.3. Let M be a compact flat surface in E^m with $\lambda_2=0$ and k a positive even integer. Then we have

$$TA_2(k)=\frac{2C_{m+2k-3}}{C_{2k+1}}\sum\frac{(-1)^s k!}{r!\,s!}\;B\left(\frac{2r+1}{2},\frac{2s+1}{2}\right)\;\int_M\lambda_1(p)^k dv,$$

where r and s are nonnegative integers such that $r+s=k$.

3. Pseudo-umbilical surfaces in E^m

Let M be a pseudo-umbilical surface in a Euclidean m-space E^m. Then we can choose a local field of orthonormal frames $e_1, e_2, \xi_3, \cdots, \xi_m$ defined along M such that e_1, e_2 are tangent, $\xi_3, \cdots, \xi_m$ are normal to M and the i-th mean curvatures $(i=1,2)$ are given by

$$K_1(p,e)=\frac{1}{2}\sum(h_{11}^r+h_{22}^r)\;\cos\theta_r,$$

$$K_2(p,e)=\sum\lambda_{r-2}(p)\;\cos^2\theta_r,\;\;\lambda_1\geqq\lambda_2\geqq\cdots\geqq\lambda_{m-2}$$

for a unit normal vector $e=\sum\cos\theta_r\xi_r$ at p in M. Since M is a pseudo-umbilical surface, if we take ξ_3 in the direction of the mean curvature vector, then

$$h_{11}^3=h_{22}^3=\alpha,\;\;h_{12}^3=h_{21}^3=0\;\text{ and }\;h_{11}^r+h_{22}^r=0.$$

for $r=4,5,\cdots,m$, where α is the mean curvature at p ([3]).

Theorem 3.1. Let M be a compact pseudo-umbilical surface in E^m and let k be any positive integer. Then the first total absolute curvature $TA_1(k)$ of order k is given by

$$TA_1(k)=\frac{\Gamma(\frac{k+1}{2})}{\Gamma(\frac{k}{2}+1)}\frac{C_{m+k-3}}{C_{k+1}}\,2\sqrt{\pi}\int_M\alpha^k dv,$$

where Γ is the Gamma function.

Proof. For a unit normal vector $e=\sum\cos\theta_r\xi_r$ at p,

since $|K_1(p,e)|=\frac{1}{2}|\sum(h^r_{11}+h^r_{22})\cos\theta_r|=\alpha|\cos\theta_3|$,

$$\begin{aligned}TA_1(k)&=\int_M\alpha^k[\int_{S^{m-3}}|\cos\theta_3|\,d\sigma]dv\\&=\frac{C_{m+k-3}}{C_{k+1}}\int_M\alpha^k[\int_0^{2\pi}|\cos\theta_3|^k d\theta_3]dv\\&=\frac{C_{m+k-3}}{C_{k+1}}\,2\sqrt{\pi}\frac{\Gamma(\frac{k+1}{2})}{\Gamma(\frac{k}{2}+1)}\int_M\alpha^k dv\end{aligned}$$

(by spherical integration ([8]).

Remark. From Theorem 3.1, we obtain

$$TA_1(2)=\frac{1}{2\pi}C_{m-1}\int_M\alpha^2 dv\geqq 2C_{m-1},\quad \text{since } \int_M\alpha^2\,dv\geqq 4\pi.$$

And, if the equality sign holds, then M is a 2-sphere in an affine 3-space ([1,I], [11]).

Theorem 3.2. Let M be a compact pseudo-umbilical surface in E^m with $\lambda_{m-2}=0$ and k a positive integer. Then we have

$$TA_2(k)=\frac{2\sqrt{\pi}\;\Gamma(k+\frac{1}{2})\,C_{m+2k-3}}{k!\;\;C_{2k+1}}\int_M\alpha^k dv.$$

Proof. Since $0\geqq\lambda_2\geqq\cdots\geqq\lambda_{m-2}=0$,

$$K_2(p,e)=\sum\lambda_{r-2}(p)\cos^2\theta_r=\lambda_1(p)\cos^2\theta_3$$

for a unit normal vector $e=\sum\cos\theta_r e_r$ at p. Hence

$$TA_2(k)=\int_M[\int_{S^{m-3}} \alpha^{2k}\cos^{2k}\theta_3 \ d\sigma]dv.$$

By spherical integration ([8]), we have

$$TA_2(k)=\frac{C_{m+2k-3}}{C_{2k+1}}\int_M\alpha^{2k}[\int_0^{2\pi}\cos^{2k}\theta_3 d\theta_3]dv$$

$$=\frac{C_{m+2k-3}}{C_{2k+1}} \ 2\sqrt{\pi}\frac{\Gamma(k+\frac{1}{2})}{\Gamma(k+1)}\int_M\alpha^{2k}dv.$$

Remark. For a compact pseudo-umbilical surface in E^m with $\lambda_{m-2}=0$,

$$TA_2(1)=\frac{\pi C_{m-1}}{C_3} \int_M\alpha^2 dv. \ =C_{m-1}\chi(M)$$

by the Gauss-Bonnet theorem, where $\chi(M)$ is the Euler characteristic of M. Since $TA_2(1)\geqq C_{m-1}\beta(M)$ for the sum $\beta(M)$ of the betti numbers of M *(see* [4,I]), $\chi(M)=\beta(M)=2$. Hence M is a 2-sphere. This is consistent with [9].

For a compact pseudo-umbilical flat surface M in E^m with flat normal connection, we can choose a local field of orthonormal frames $e_1,e_2,\cdots,e_m$ such that

$$h^3_{11}=h^3_{22}=h^4_{11}=-h^4_{22}=\alpha,\ h^3_{12}=h^4_{12}=h^r_{ij}=0$$

for $r=5,6,\cdots,m$ ([3]). For a positive even integer k,

$$TA_2(k)=\int_M\alpha^{2k}[\int_{S^{m-3}}|\cos^2\theta_3-\cos^2\theta_4|^k d\sigma]dv$$

$$=\int_M\alpha^{2k}\sum\frac{(-1)^s k!}{r!\,s!} \ [\int_{S^{m-3}}\cos^{2r}\theta_3\cos^{2s}\theta_4 d\sigma]dv$$

$$=2\frac{C_{m+2k-3}}{C_{2k+1}}\sum\frac{(-1)^s k!}{r!\,s!} \quad B(\frac{2r+1}{2},\frac{2s+1}{2})\int_M\alpha^{2k}dv$$

for nonnegative integers r,s such that $r+s=k$. And

$$TA_2(1)=\int_M \alpha^2[\int_{S^{m-3}} |\cos^2\theta_3-\cos^2\theta_4|\,d\sigma]dv$$
$$=\frac{2C_{m-1}}{\pi^2}\int_M \alpha^2 dv \ \geqq 4\pi C_{m-1}.$$

The equality sign holds if and only if M is a Clifford torus, i.e, M is the product surface of two plane circles with the same radius ([3]).

Hence we obtain the following theorem.

Theorem 3.3. Let M be a compact flat pseudo-umbilical surface in E^m with flat normal connection and let k be a positive even integer. Then we have

$$TA_2(k)=\frac{2C_{m+2k-3}}{C_{2k+1}}\sum\frac{(-1)^s k!}{r!\,s!}\ B\ (\frac{2r+1}{2},\frac{2s+1}{2})\ \int_M \alpha^{2k}dv$$

for nonnegative integers r,s such that $r+s=k$.

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National Fisheries University
Pusan 608−737, Korea