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CERTAIN CLASSES OF UNIVALENT FUNCTIONS WITH NEGATIVE COEFFICIENTS

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In this paper, we define new classes $S^*(\alpha,\beta,\gamma)$ and $C^*(\alpha,\beta,\gamma)$ of T, the class of analytic and univalent functions with negative coefficients. We have sharp results concerning coefficients, distortion of functions belonging to these classes along with a representation formular for the function in $S^*(\alpha,\beta,\gamma)$ and $C^*(\alpha,\beta,\gamma)$. Furthermore, we improve the results of Libera for the class of starlike functions having negative coefficients.

1. Introduction.

Let S denote the class of functions of the form

$$f(z)=z+\sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

which analytic and univalent in the unit disk $U=\{z: |z|<1\}$. A function $f(z)$ in the class S is said to be starlike of order $\delta(0\leq\delta<1)$, denoted by $f(z)\in S(\delta)$, if $Re\ \{zf'(z)/f(z)\}>\delta$ ($|z|<1$) and it is said to be convex of order δ, denoted by $f\in C(\delta)$, if $Re\ \{1+zf''(z)/f'(z)\}>\delta$ ($|z|<1$).

Let $S(\alpha,\beta,\gamma)$ denote the class of the functions defined by (1.1) that are analytic in the unit disk U and

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$$(1.2)\qquad \left|\frac{zf'(z)/f(z)-1}{\alpha zf'(z)/f(z)+(1-\gamma)}\right|<\beta \qquad (|z|<1)$$

for some $\alpha(0\leq\alpha\leq 1)$, $\beta(0<\beta\leq 1)$, $\gamma(0\leq\gamma<1)$.

We note that $S(\alpha,\beta,\gamma)$ is a subclass of $S(0)$ and also a subclass of S.

Furthermore, $f(z)\in C(\alpha,\beta,\gamma)$ if and only if $zf'(z)\in S(\alpha,\beta,\gamma)$. In particular, the classes $S(1,\beta,0)$ and $S(0,1,0)$ are studied Padamanabhan [9] and Singh [11, 12], respectively.

Let T denote the class of functions which are analytic and univalent in U with the form

$$(1.3)\qquad f(z)=z-\sum_{n=2}^{\infty}|a_n|z^n.$$

We denote that

$$S^*(\delta)=S(\delta)\cup T,\quad S^*(\alpha,\beta,\gamma)=S(\alpha,\beta,\gamma)\cap T \text{ and } C^*(\alpha,\beta,\gamma)=C(\alpha,\beta,\gamma)\cap T.$$

In particular $S^*(1,\beta,\gamma)$ and $C^*(1,\beta,\gamma)$ $(0<\beta\leq 1,\ 0\leq\gamma\leq 1/2)$ are studied by Gupta and Jain [2], Juneja and Mogra [4]. Also Silverman [10] determined the coefficient inequalities, distortion and covering theorems for the classes $S^*(0,1,\delta)=S^*(\delta)$.

In this paper, the sharp results concerning coefficients and distortion theorems for the classes $S^*(\alpha,\beta,\gamma)$ and $C^*(\alpha,\beta,\gamma)$ are determined.

Furthermore we improve the results of Libera for the class of starlike functions having negative coefficients. It is shown that the class $S^*(\alpha,\beta,\gamma)$ is closed under "convex linear combination".

2. Coefficients theorems

Theorem 2.1. The function $f(z)$ defined by (1.3) is in the class $S^*(\alpha,\beta,\gamma)$ if and only if

$$(2.1)\qquad \sum_{n=2}^{\infty}\{(n-1)+\beta(\alpha n+1-\gamma)\mid a_n\mid\leq\beta(\alpha+1-\gamma).$$

The result is sharp.

Proof. Let $|z|=1$. Then we have

$$\begin{aligned}
&|zf'(z)-f(z)|-\beta|\alpha zf'(z)+(1-\gamma)f(z)|\\
&=|\sum_{n=2}^{\infty}(n-1)|a_n|z^n|-\beta|(\alpha+1-\gamma)z\\
&\quad-\sum_{n=2}^{\infty}(\alpha n+1-\gamma)|a_n|z^n|\\
&\leq|\sum_{n=2}^{\infty}(n-1)+\beta(\alpha n+1-\gamma)\}|a_n|z^n|-\beta(+1-\gamma)|z|\\
&\leq\sum_{n=2}^{\infty}\{(b-1)+\beta(\alpha n+1-\gamma)\}|a_n|-\beta(\alpha+1-\gamma)\\
&\leq 0.
\end{aligned}$$

Hence, by the maximum modulus theorem, f is in the class $S^*(\alpha,\beta,\gamma)$.

For the converse, assume that

$$(2.2)\qquad \frac{zf'(z)/f(z)-1}{\alpha zf'(z)/f(z)+(1-\gamma)}$$

$$=\frac{|\sum_{n=2}^{\infty}(n-1)|a_n|z^n|}{|(\alpha+1-\gamma)z-\sum_{n=2}^{\infty}(\alpha n+1-\gamma)|a_n|z^n|}<\beta.$$

From (2.2), we have

$$(2.3)\qquad Re\{\frac{\sum_{n=2}^{\infty}(n-1)|a_n|z^n}{(\alpha+1-\gamma)z-\sum_{n=2}^{\infty}(\alpha n-1-\gamma)|a_n|z^n}<\beta.$$

Choose values of z on the real axis so that $zf'(z)/f(z)$ is real. Upon clearing the denominator in (2.3) and letting $z \longrightarrow 1$ through real values, we obtain

$$\sum_{n=2}^{\infty}(n-1)\mid a_n\mid \leq \beta\{(\alpha+1-\gamma)z-\sum_{n=2}^{\infty}(\alpha n+1-\gamma)-a_n\mid\}.$$

This completes the proof of the theorem.

Finally, the function

$$f(z)=z-\frac{\beta(1-\gamma+\alpha)}{1+\beta(1-\gamma+2\alpha)}z^2 \tag{2.4}$$

is an extremal function.

Corollary 2.2. If the function $f(z)$ defined by (1.3) belongs to the class $S^*(\alpha,\beta,\gamma)$, then

$$\mid a_n\mid \leq \frac{\beta(\alpha+1-\gamma)}{(n-1)+\beta(\alpha n+1-\gamma)} \qquad (n=2,3,\ldots). \tag{2.5}$$

Corollary 2.3. [8, 10] If the function $f(z)$ defined by (1.3) belongs to the class $S^*(\delta)$, then

$$\mid a_n\mid \leq \frac{1-\delta}{n-\delta} \qquad (n=2,3,\ldots). \tag{2.6}$$

Proof. We note that $S^*(\delta)=S^*(0,1,\delta)$.

The following theorem 2.4 imediately follows by appealing to theorem 2.1.

Theorem 2.4. The function $f(z)$ defined by (1.3) is in the class $C^*(\alpha,\beta,\gamma)$ if and only if

$$\sum_{n=2}^{\infty} n\{(n-1)+\beta(\alpha n+1-\gamma)\}\mid a_n\mid \leq \beta(\alpha+1-\gamma). \tag{2.7}$$

Corollary 2.5. If the function $f(z)$ defined by (1.3) belongs to the class $C^*(\alpha,\beta,\gamma)$, then

$$(2.8) \qquad |a_n| \leq \frac{\beta(\alpha+1-\gamma)}{n[(n-1)+\beta(\alpha n+1-\gamma)]} \qquad (n=2,3,\ldots).$$

3. Distortion Theorems

Theorem 3.1. If the function $f(z)$ defined by (1.3) belongs to the class $S^*(\alpha,\beta,\gamma)$, then for $|z|=r$

$$(3.1) \qquad \frac{r-\beta(\alpha+1-\gamma)}{1+\beta(2\alpha+1-\gamma)} r^2 \leq |f(z)|$$

$$\leq \frac{r+\beta(\alpha+1-\gamma)}{1+\beta(2\alpha+1-\gamma)} r^2.$$

and

$$(3.2) \qquad \frac{1-2\beta(\alpha+1-\gamma)}{1+\beta(2\alpha+1-\gamma)} r \leq |f'(z)|$$

$$\leq \frac{1+2\beta(\alpha+1-\gamma)}{1+\beta(2\alpha+1-\gamma)} r.$$

The equalities are attained for the functions

$$f(z)=z-\frac{\beta(\alpha+1-\gamma)}{1+\beta(2\alpha+1-\gamma)} z^2.$$

Proof. From (2.1), we note that

$$(3.3) \qquad 1+\beta(2\alpha+1-\gamma) \sum_{n=2}^{\infty} |a_n| \leq \beta(\alpha+1-\gamma).$$

This gives us

$$(3.4)\qquad \sum_{n=2}^{\infty} |a_n| \leq \frac{\beta(\alpha+1-\gamma)}{1+\beta(2\alpha+1-\beta)}$$

Therefore

$$(3.5)\qquad |f(z)| \leq |z| + \sum_{n=2}^{\infty} |a_n| \; |z|^n$$

$$\leq r + \frac{\beta(\alpha+1-g)}{1+\beta(2\alpha+1-\gamma)} r^2$$

and

$$|f(z)| \geq |z| - \sum_{n=2}^{\infty} |a_n| \; |z|^n$$

$$\geq r - \frac{\beta(\alpha+1-\gamma)}{1+\beta(\alpha+1-\gamma)} r^2$$

Thus (3.1) follows. Furthermore, by Theorem (2.1) we have

$$(3.6)\qquad \sum_{n=2}^{\infty} n\, |a_n| \leq \frac{2\beta(\alpha+1-\gamma)}{1+\beta(2\alpha+1-\gamma)}$$

Hence

$$(3.7)\qquad |f'(z)| \leq 1 + \sum_{n=2}^{\infty} n\, |a_n| \; |z|^n$$

$$\leq 1 + r \sum_{n=2}^{\infty} n\, |a_n|$$

$$\leq 1 + \frac{2\beta(\alpha+1-\gamma)}{1+\beta(2\alpha+1-\gamma)} r$$

and

$$(3.8)\qquad |f'(z)| \geq 1-\sum_{n=2}^{\infty} n\,|a_n|\,|z|^n$$

$$\geq 1-r\sum_{n=2}^{\infty} n\,|a_n|$$

$$\geq 1-\frac{2\beta(\alpha+1-\gamma)}{1+\beta(2\beta+1-\gamma)}\,r.$$

Using same technique as in the proof of Theorem 3.1, we obtain the followings from Theorem 2.2.

Theorem 3.2. If the function $f(z)$ defined by (1.3) is in the class $C^*(\alpha,\beta,\gamma)$, then then for $|z|=r$

$$(3.9)\qquad r-\frac{\alpha+1-\gamma}{2(2\alpha+1-\gamma)}r^2 \leq |f(z)|$$

$$\leq r+\frac{\alpha+1-\gamma}{2(2\alpha+1-\gamma)}r^2$$

and

$$(3.10)\qquad 1-\frac{\alpha+1-\gamma}{2\alpha+1-\gamma}r \leq |f'(z)|$$

$$\leq 1+\frac{\alpha+1-\gamma}{2\alpha+1-\gamma}r$$

The equalities are ettained for the functions

$$(3.11)\qquad f(z)=z-\frac{\alpha+1-\gamma}{2(2\alpha+1-\gamma)}z^2$$

From Theorem 3.1 and Theorem 3.2, we get the Corollary 3.3. and Corollary 3.4.

Corollary 3.3. If the function $f(z)$ is in the class $S^*(\alpha,\beta,\gamma)$, then the disk $|z|<1$ is mapped onto a domain that containes the disk $|w|<(1+\alpha\beta)/\{1+\beta(2\alpha+1-\gamma)\}$ under f. The result is sharp with

$$(3.12)\qquad f(z)=z-\frac{2\beta(1-\alpha)}{1+\beta(3-2\alpha)}z^2$$

Corollary 3.4. Of the function $f(z)$ is in the class $C^*(\alpha,\beta,\gamma)$, then the $|z|<1$ is mapped onto a domain that contains the disk $|w|<(3\alpha+1-\gamma)/2(2\alpha+1-\gamma)$ under f. The result is sharp with the exremal function

$$(3.13)\qquad f(z)=z-\frac{\alpha+1-\gamma}{2(2\alpha+1-\gamma)}z^2.$$

4. Extreme points for $S^*(\alpha,\beta,\gamma)$ and $C^*(\alpha,\beta,\gamma)$.

In view of Theorem 2.1, the class $S^*(\alpha,\beta,\gamma)$ and $C^*(\alpha,\beta,\gamma)$ is closed under convex linear combinations.

In [1], Brickman, MacGregor and Wilkin found the extrem points of a closed convex hull for convex, starlike, close-to-convex, and typically real functions. Since then, the extreme points for many additional classes have been determined. We shall now determine the extreme points of $S^*(\alpha,\beta,\gamma)$.

Theorem 4.1. Let $f(z)=z$ and

$$(4.1)\qquad f_n(z)=z-\frac{\beta(\alpha+1-\gamma)}{(n-1)+\beta(\alpha n+1-\gamma)}z^2\quad (n=2,3,\dots)$$

Then $f(z)$ is in the class $S^*(\alpha,\beta,\gamma)$ if and only if

$$(4.2)\qquad f(z)=\sum_{n=2}^{\infty}\tau_n f_n(z)$$

where $\tau_n \geq 0$ *and* $\sum_{n=2}^{\infty} \tau_n = 1$.

Proof. Suppose $f(z) = \sum_{n=2}^{\infty} \tau_n f_n(z)$

$$= z - \sum_{n=2}^{\infty} \tau_n \frac{\beta(\alpha+1-\gamma)}{(n-1)+\beta(\alpha n+1-\gamma)} z^2$$

Then

$$(4.3) \qquad \sum_{n=2}^{\infty} \tau_n \frac{\beta(\alpha+1-\gamma)}{(n-1)+\beta(\alpha n+1-\gamma)} \quad \frac{(n-1)+\beta(\alpha n+1-\gamma)}{\beta(\alpha+1-\gamma)}$$

$$= \sum_{n=2}^{\infty} \tau_n = 1 - \tau_1 < 1.$$

Thus $f(z)$ is in the class $S^*(\alpha,\beta,\gamma)$.

Conversely, suppose $f(z)$ is in the class $S^*(\alpha,\beta,\gamma)$. Since

$$(4.4) \qquad |a_n| \leq \frac{\beta(\alpha+1-\gamma)}{(n-1)+\beta(\alpha n+1-\gamma)} \quad (n=2,3,\ldots),$$

we may set

$$(4.5) \qquad \tau_n = \frac{(n-1)+\beta(\alpha n+1-\gamma)}{\beta(\alpha+1-\gamma)} a_n \qquad (n=2,3,\ldots)$$

and

$$(4.6) \qquad \tau_1 = 1 - \sum_{n=2}^{\infty} \tau_n.$$

Then $f(z) = \sum_{n=2}^{\infty} \tau_n f_n(z)$.

Corollary 4.2. The extreme points of $S^*(\alpha,\beta,\gamma)$ are the functions $f_n(z)(n=2,3,\ldots)$, defined in Theorem 4.1.

Corollary 4.3. Let $f_1(z)=z$ and

$$(4.6)\qquad z_n(z)=z-\frac{\beta(\alpha+1-\gamma)}{n[(n-1)+\beta(\alpha n+1-\gamma)]}-z^n\quad(n=2,3,\ldots)$$

Then $f(z)$ is in the class $C^*(\alpha,\beta,\gamma)$ if and only if

$$(4.7)\qquad f(z)=\sum_{n=1}^{\infty}\tau_n f_n(z),$$

where $\tau_n\geq 0$ *and* $\sum_{n=1}^{\infty}\tau_n=1.$

The extreme points of $C^*(\alpha,\beta,\gamma)$ are the above functions $f_n(z)$.

5. Integral Operators

In [5], Libera consider an integral operator and Livingston [6], Kumer and Shukla [7] investigate the integral operator. In [3], Bernardi consider the generalized integral operator, and investigate the interesting results. The following theorems are the results for the generalized integral operator.

In order to prove our theorems in this chaper, we have to recall here the following results due to Silverman [10], McCarty [8].

Lemma 5.1. If a function $f(z)$ defined by (1.3) belongs to the class $S^*(\delta)$, if and oly if

$$(5.1)\qquad \sum_{n=2}^{\infty}\{(n-\delta)/(1-\delta)\}\mid a_n\mid\leq 1$$

for some $\delta(0\leq\delta<1)$.

Theorem 5.2. Let c be a real number such that $c>-1$. If a function $f(z)$ defined by (1.3) belongs to the class $S^*(\alpha,\beta,\gamma)$, then the function defined by

(5.2) $$F(z)=\frac{c+1}{z}\int_0^z t^{c-1}f(t)\ dt$$

also belong to $S^*(\alpha,\beta,\gamma)$.

Proof. From the representation or $F(z)$, we have

(5.3) $$F(z)=z-\sum_{n=2}^{\infty}|b_n|z^n=z-\sum_{n=2}^{\infty}\frac{c+1}{c+n}|a_n|z^n.$$

Hence

(5.4) $$\sum_{n=2}^{\infty}[(n-1)+\beta(\alpha n+1-\gamma)]|b_n|$$
$$=\sum_{n=2}^{\infty}\{(n-1)+\beta(\alpha n+1-\gamma)\}\frac{c+1}{c+n}|a_n|.$$
$$\leq\sum_{n=2}^{\infty}\{(n+1)+\beta(\alpha n+1-\gamma)\}|a_n|$$
$$\leq\beta(\alpha+1-\gamma),$$

Since $f(z)\in S^*(\alpha,\beta,\gamma)$. By Theorem 1.1, $F(z)$ belongs to the class $S^*(\alpha,\beta,\gamma)$.

Corollary 5.3. Let c be a real number such that $c>-1$. If a function $f(z)$ defined by (1.3) belongs to the class $C^*(\alpha,\beta,\gamma)$, then the function defined by (5.2) also belongs to $C^*(\alpha,\beta,\gamma)$.

Theorem 5.4. If a function $f(z)$ defined by (1.3) belongs to the class $S^*(\alpha,\beta,\gamma)$, then the function $F(z)$ defined by (5.2) belongs to the class $S^*(\delta)$, where

(5.5) $$\delta=\frac{(c+2)\{1+\beta(2\alpha+1-\gamma)\}-2\beta(c+1)(\alpha+1-\gamma)}{(c+2)\{1+\beta(2\alpha+1-\gamma)\}-\beta(c+1)(\alpha+1-\gamma)}<1.$$

Proof. From Theorem 2.1, we note that

$$(5.6)\qquad \sum_{n=2}^{\infty}\left\{\frac{(n-1)+\beta(\alpha n+1-\gamma)}{\beta(\alpha+1-\gamma)}\right\}|a_n|\leq 1.$$

From Lemma 5.1, if $F(z)\in S^*(\delta)$, then

$$(5.7)\qquad \sum_{n=2}^{\infty}\frac{n-\delta}{1-\delta}|b_n|\leq 1.$$

The above inequality (5.7) is true if

$$(5.8)\qquad \frac{n-\delta}{1-\delta}|b_n|\leq\frac{(n-1)+\beta(\alpha n+1-\gamma)}{\beta(\alpha+1-\gamma)}|a_n|$$

for each $(n=1,\ 2,\ldots.)$, which is equivalent to

$$(5.9)\qquad \delta\leq\frac{(c+n)\{(n-1)+\beta(\alpha n+1-\gamma)\}-n\beta(c+1)(\alpha+1-\gamma)}{(c+n)\{(n-1)+\beta(\alpha n+1-\gamma)\}-\beta(c+1)(\alpha+1-\gamma)}$$

$$=\delta_n.$$

It is easy to verify that δ_n is an increasing function of n. Therefore $\delta=\inf_{n\geq 2}\delta_n=\delta_2$. From this fact, we obtain

$$(5.10)\qquad \delta=\frac{(c+2)\{1+\beta(2\alpha+1-\gamma)\}-2\beta(c+1)(\alpha+1-\gamma)}{(c+2)\{1+\beta(2\alpha+1-\gamma)\}-\beta(c+1)(\alpha+1-\gamma)}$$

The following corollary is the result of Libera [5, Theorem 1] for the class of starlike funtions having negative coefficients.

Corollary 5.5. If a function $f(z)$ defined by (1.3) belongs to the class $S^*(0)\equiv S^*$ then the function $F(z)$ defined by (5.2) belongs to the class $S^*(1/2)$. The result is sharp. The converse need not be true.

Theorem 5.6. If a function $f(z)$ defined by (1.3) belons to the class $C^*(\alpha,\beta,\gamma)$, then the function $F(z)$ defined by (5.2) belongs to the class $S^*(\delta)$, where

$$(5.11)\quad \delta=\frac{2(c+2)\{1+\beta(2\alpha+1-\gamma)\}-2\beta(c+1)(\alpha+1-r)}{2(c+2)\{1+\beta(2\alpha+1-r)\}-\beta(c+1)(\alpha+1-\gamma)}$$

Theorem 5.7. Let $F(z)=z-\sum_{n=2}^{\infty}|a_n|z^n$. If $F(z)$ is in the class $S^*(\alpha,\beta,\gamma)$, then the function $f(z)$ defined by (5.2) belongs to the class $S^*(\delta)$ in $|z|<r(\delta;\alpha,\beta,\gamma)$, where

$$r(\delta;\alpha,\beta,\gamma)=\inf_{n\geq 2}\left\{\frac{1-\delta}{n-\delta}\frac{n}{n+1}\ \frac{(n-1)+\beta(\alpha n+1-\gamma)}{\beta(\alpha+1-\gamma)}\right\}^{\frac{1}{n-1}}$$

The result is sharp.

Proof. Since $F(z)=z-\sum_{n=2}^{\infty}|a_n|z^n$, it follows from (6.1) that

$$(5.12)\quad f(z)=z-\sum_{n=2}^{\infty}\frac{n+1}{2}|a_n|z^n.$$

In orther to establish the required result, it suffices to show that

$$(5.13)\qquad \left|\frac{zf'(z)}{f(z)}-1\right|<1-\delta$$

in $|z|<r(\delta;\alpha,\beta,\gamma)$. Now

$$(5.14)\qquad \left|\frac{zf'(z)}{f(z)}-1\right| =\left|\frac{z-\sum_{n=2}^{\infty}\frac{n(n+1)}{2}|a_n|z^n}{z-\sum_{n=2}^{\infty}\frac{n+1}{2}|a_n|z^n}-1\right|$$

$$= \left| \frac{-\sum_{n=2}^{\infty} \frac{(n-1)(n+1)}{2} \, |a_n| \, |z|^{n-1}}{1-\sum_{n=2}^{\infty} \frac{n+1}{2} \, |a_n| \, |z|^{n-1}} \right|$$

$$\leq \frac{\sum_{n=2}^{\infty} (n-1) \frac{n+1}{2} \, |a_n| \, |z|^{n-1}}{1-\sum_{n=2}^{\infty} \frac{n+1}{2} \, |a_n| \, |z|^{n-1}}$$

$$< 1-\delta$$

provided

(5.15) $$\sum_{n=2}^{\infty} \frac{n-\delta}{1-\delta} \, \frac{n+1}{2} \, |a_n| \, |z|^{n-1} < 1.$$

But, for $F(z) \in S^*(\alpha,\beta,\gamma)$, Theorem 2.1 ensures that

(5.16) $$\sum_{n=2}^{\infty} \frac{(n-1)+\beta(\alpha n+1-\gamma)}{\beta(\alpha+1-\gamma)} \, |a_n| \leq 1$$

Hence, the inequality (5.15) holds if

(5.17) $$\frac{n-\delta}{1-\delta} \, \frac{n+1}{2} \, |z|^{n-1} < \frac{(n+1)+\beta(\alpha n+1-\gamma)}{\beta(\alpha+1-\gamma)}$$

for each $n=2, 3, \ldots$, or if

(5.18) $$|z| < \left\{ \frac{1-\delta}{n-\delta} \, \frac{2}{n-1} \, \frac{(n-1)+\beta(\alpha n+1-\gamma)}{\beta(\alpha+1-\gamma)} \right\}^{\frac{1}{n-1}}$$

for each $n=2, 3, \ldots$ Hence, $f(z) \in S^*(\delta)$ in $|z| < r(\delta;\alpha,\beta,\gamma)$. Sharpness follows if we take the function $F(z)$ given by

$$(5.19)\quad F(z)=z-\frac{\beta(\alpha+1-\gamma)}{(n-1)+\beta(\alpha n+1-\gamma)}z^n,\quad n=2,\ 3,\ldots$$

Corollary 5.8. Let $F(z)=z-\sum_{n=2}^{\infty}|a_n|z^n$. If $F(z)$ is in the class $C^*(\alpha,\beta,\gamma)$, then the function $f(z)$ defined by (5.2) belongs to the class $S^*(\delta)$ in $|z|<r(\delta;\alpha,\beta,\gamma)$, where

$$r(\delta;\alpha,\beta,\gamma)=\inf_{n\geq 2}\left\{\frac{1-\delta}{n-\delta}\,\frac{n}{n+1}\,\frac{n[(n-1)+\beta(\alpha n+1-\gamma)]}{\beta(\alpha+1-\gamma)}\right\}^{\frac{1}{n-1}}$$

The result is sharp.

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