Pusan Kyŏngnam Mathematical Journal
Vol 5, 125~134, 1989

SEVERAL CONTINUITIES OF FUNCTIONS ON CONVERGENCE SPACES*

Bu Young Lee

1. Introduction

The purpose of the present note is to introduce some new classes of functions on convergence spaces. These classes contain the class of continuous functions. For topological spaces, Levine. Noiri. Papp, Reilly, Vamanamurthy, and Singal ([8], [9], [10], [11], [12], and [13]) investigated some properties of these classes.

Kent ([4]) introduced a convergence structure on a nonempty set. Kent and Richardson ([5], [6], and [7]) investigated some properties of convergence spaces.

For a nonempty set S, $F(S)$ denotes the set of all filters and $P(S)$ the set of all subsets of S. For each $s \in S$, $\dot{s}$ is the principal ultrafilter containing $\{s\}$.

A convergence structure on S is defined to be a function q from $F(S)$ into $P(S)$, satisfying the following conditions :

(1) for each $s \in S$, $s \in q(\dot{s})$;

(2) if ψ and ϕ are in $F(S)$ and $\psi \supset \phi$, then $q(\psi) \supset q(\phi)$,

(3) if $s \in q(\psi)$, then $s \in q(\psi \cap \dot{s})$

The pair (S,q) is called a convergence space. If $s \in q(\psi)$, then we say that ψ q-converges to s. The filter $V_q(s)$ obtained by intersecting all filters which q-converges to s is called the *q-neighborhood filter* at s. If $V_q(s)$ q-converges to s for each $s \in S$, then (S,q) is called a pretopological space.

Let $C(S)$ be the set of all convergence structures on S, partially ordered as follows : $q_1 \leq q_2$ if and only if $q_1(\psi) \supset q_2(\psi)$, for all $\psi \in F(S)$.

Received May. 30. 1989
*Research supported by a grant from the Ministry of Education, 88-89

The set function $\Gamma_q:P(S)\to P(S)$ is defined on all subset $A\subset S$ by

$$\Gamma_q(A)=\{s\in S \mid \text{there exists an ultrafilter } \psi \text{ } q\text{-converging to } s \text{ with } A\in\psi\}.$$

Γ_q is a closure operator in the topological sense, except idempotency.

The set function $I_q.P(S)\to P(S)$ defined recursively for every subset A of S as follows :

$$I_q(A)=\{s\in A \mid A\in V_q(s)\}$$

I_q has all of the properties of a topological interior operator except idempotency.

If ψ is any filter on S, $\Gamma_q(\psi)$ $(I_q(\psi))$ is the filter generated by $\{\Gamma_q(F) \mid F\in\psi\}$ $(\{I_q(F) \mid F\in\psi\})$.

A function f, mapping a convergence space (S,q) onto a convergence space (T,p), is said to be *continuous* if $f(\psi)$ p-converges to $f(s)$ whenever ψ q-converges to s.

Let X be a nonempty set, (X_λ,q_λ) a convergence space, and for each $\lambda\in\Lambda$, f_λ a function X onto (X_λ,q_λ). The *initial convergence structure* q on X induced by the family $\{f_\lambda \mid \lambda\in\Lambda\}$ is defined to be a function from $F(X)$ into $P(X)$ satisftying the following condition :

for each $x\in X$ and $\psi\in F(X)$, $x\in q(\psi)$ if and only if $f_\lambda(\psi)$ q_λ-converges to $f_\lambda(x)$ for each $\lambda\in\Lambda$.

Throughout the present paper, spaces always mean convergence spaces and $(\prod_{\lambda\in\Lambda}X_\lambda,q')$ the initial convergence space induced by the family $\{P_\lambda \mid \lambda\in\Lambda,\ P_\lambda\cdot\prod_{\lambda\in\Lambda}X_\lambda\to X_\lambda$ is the canonical projection$\}$. $(\prod_{\lambda\in\Lambda}X_\lambda,q')$ is called the *product space* of a family $\{(X_\lambda,q_\lambda) \mid \lambda\in\Lambda\}$ of convergence spaces.

A space (S,q) is said to be *regular* if $\Gamma_q(\psi)$ q-converges to s whenever ψ q-converges to s.

2. θ-continuous functions

A function f from a space (S,q) onto another space (T,p) is said to be *θ-continuous* at a point s in S if $\Gamma_p(V_p(f(s)))\subset f(\Gamma_q(\psi))$ whenever ψ q-converges to s. f is said to be *θ-continuous* if f is θ-continuous at each $s\in S$.

Clearly, continuous functions are always θ-continuous.

The following properties are easily verified.

Theorem 2.1. *Let (T,p) be a pretopological space If $f: (S,q)\to(T,p)$ and $g:(T,p)\to(R,r)$ are θ-continuous, then $g \cdot f$ is so.*

Theorem 2.2. *Let $f:(S,q)\to(T,p)$ be θ-continuous If (T,p) is a pre-topological regular space, then f is continuous.*

Lemma 2.3. Let $(\prod_{\lambda\in\Lambda} X_\lambda, q')$ *be the prouct space of a family $\{(X_\lambda, q_\lambda)\mid\lambda\in\Lambda\}$ of spaces. Then for each $x=(x_\lambda)_{\lambda\in\Lambda}\in\prod_{\lambda\in\Lambda} X_\lambda$,*

(1) $\prod_{\lambda\in\Lambda} V_{q_\lambda}(x_\lambda)\subset V_{q'}(x)$,

(2) $P_\lambda(V_{q'}(x))=V_{q_\lambda}(x_\lambda)$ *for each $\lambda\in\Lambda$, where P_λ is the canonical projection.*

Proof. (1) Let F be any element of $\prod_{\lambda\in\Lambda} V_{q_\lambda}(x_\lambda)$. Then for each $\lambda\in\Lambda$, there exists $F_\lambda\in V_{q_\lambda}(x)$ such that $\prod_{\lambda\in\Lambda} F_\lambda\subset F$ Suppose that $\prod_{\lambda\in\Lambda} F_\lambda\notin V_{q'}(x)$, then there exists a filter ψ q-converging to x such that $\prod_{\lambda\in\Lambda} F_\lambda\in\psi$. Therefore $P_\lambda(\psi)$ q_λ*-converges to x_λ for each* $\lambda\in\Lambda$ and there is a $\mu\in\Lambda$ such that $F_\mu\notin P_\mu(\psi)$. But $F_\lambda\in V_{q_\lambda}(x_\lambda)$. Thus $F\in V_{q'}(x)$.

(2) Since

$$\prod_{\lambda\in\Lambda} V_{q_\lambda}(x_\lambda)\subset V_{q'}(x),\ V_{q_\lambda}(x_\lambda)\subset P_\lambda(V_{q'}(x))$$

for each $\lambda\in\Lambda$ Let G_μ be any element of $P_\mu(V_{q'}(x))$ for each $\mu\in\Lambda$, then there is $F\in V_{q'}(x)$ such that $P_\mu(F)\subset G_\mu$. Let ψ_λ be a filter which q_λ-converges to x_λ for each $\lambda\neq\mu$ Then $\prod_{\lambda\in\Lambda}\psi_\lambda$ q'-converges to $x=(x_\lambda)_{\lambda\in\Lambda}$. Since $F\in V_{q'}(x)$, $F\subset\prod_{\lambda\in\Lambda} G_\lambda$. Thus $P_\mu(F)\in\psi_\mu$ and $G_\mu\in\psi_\mu$. Therefore $G_\mu\in V_{q_\mu}(x_\mu)$

Lemma 2.4 *([7]) If $f:(S,q)\to(T,p)$ is continuous and $A\subset S$, then $f(\Gamma_q(A))\subset\Gamma_p(f(A))$.*

Theorem 2.5. *If $f:(S,q)\to(\prod_{\lambda\in\Lambda} X_\lambda, q')$ is θ-continuous, then $P_\lambda\cdot f$ is θ-continuous for each $\lambda\in\Lambda$.*

Proof. For each $s\in S$, let ψ q-converge to s, then, by Lemma 2.3 and 2.4,

$$\Gamma_{q'}(V_{q'}(f(s)))\subset f(\Gamma_q(\psi)),$$

$$\Gamma_{q_\lambda}(V_{q_\lambda}(P_\lambda(f(s))))=\Gamma_{q_\lambda}(P_\lambda(V_{q'}(f(s))))$$

$$\subset P_\lambda(\Gamma_{q'}(V_{q'}(f(s))))\subset P_\lambda(f(\Gamma_q(\psi))).$$

Thus $P_\lambda \cdot f$ is θ-continuous.

Corollary 2.6. *Let $f:(S,q)\to(T,p)$ be a function and $g:(S,q)\to(S\times T, r)$ given by $g(s)=(s,f(s))$, be its graph, where r is the product convergence structure of q and p. If g is θ-continuous, then f is θ-continuous.*

A function f from a space (S,q) onto a space (T,p) is said to be *strongly θ-continuous* at a point s in S if $V_p(f(s))\subset f(\Gamma_q(\psi))$ whenever ψ q-converges to s. f is called *strongly θ-continuous* if it is strongly θ-continuous at each $s\in S$

Strongly θ-continuous functions are always θ-continuous.

Lemma 2.7 ([5]). *If $f:(S,q)\to(T,p)$ is continuous, then $f(V_q(s))\subset V_p(f(s))$ for all s in S.*

Theorem 2.8. *Let $f:(S,q)\to(T,p)$ be strongly θ-continuous.*

(1) *If (T,p) is a pretopological space, then f is contonuous.*

(2) *If $g:(T,p)\to(R,r)$ is continuous, then $g\cdot f$ is strongly θ-continuous.*

Proof. (1) Clear.

(2) Let ψ q-converge to s, then $V_p(f(s))\subset f(\Gamma_q(\psi))$ and $g(V_p(f(s)))\subset g(f(\Gamma_q(\psi)))$. Since g is continuous, by Lemma 2.7, $V_r(g(f(s)))\subset g(V_p(f(s)))$. Thus $g\cdot f$ is strongly θ-continuous.

Theorem 2.9. *Let $f:(S\ q)\to(\prod_{\lambda\in\Lambda} X_\lambda, q')$ be a function. Then f is strongly θ-continuous if and only if $P_\lambda \cdot f$ is strongly θ-continuous for each $\lambda\in\Lambda$.*

Proof. Let ψ q-converge to s, then $V_{q'}(f(s))\subset f(\Gamma_q(\psi))$. For each $\lambda\in\Lambda$, since P_λ is continuous,

$$V_{q_\lambda}(P_\lambda(f(s)))\subset P_\lambda(V_{q'}(f(s)))\subset P_\lambda(f(\Gamma_q(\psi))).$$

Thus $P_\lambda \cdot f$ is strongly θ-continuous.

Conversely, let $P_\lambda \cdot f$ be strongly θ-continuous for each $\lambda \in \Lambda$ and ψ q-converge to s, then $V_{q_\lambda}(P_\lambda f(s)) \subset P_\lambda(f(\Gamma_q(\psi)))$. Suppose that there is $N \in V_{q'}(f(s))$ such that $N \notin f(\Gamma_q(\psi))$. Then for each $F \in \psi$, since $f(\Gamma_q(F)) \not\subset N$, there exists $\mu \in \Lambda$ such that $P_\mu(f(\Gamma_q(F))) \not\subset P_\mu(N)$. And

$$P_\mu(V_{q'}(f(s))) = V_{q_\mu}(P_\mu(f(s))),$$

$$P_\mu(N) \in V_{q_\mu}(P_\mu(f(s))).$$

Therefore $P_\mu(N) \in P_\mu(f(\Gamma_q(\psi)))$. Thus there exists $F \in \psi$ such that $P_\mu(f(\Gamma_q(F))) \subset P_\mu(N)$.

Corollary 2.10. *Let $f:(S,q) \to (T,p)$ be a function and $g:(S,q) \to (S \times T, r)$ given by $g(s)=(s,f(s))$, be its graph Then f is strongly θ-continuous if and only if g is strongly θ-continuous*

Let $(\prod_{\lambda\in\Lambda} X_\lambda, q')$(resp. $(\prod_{\lambda\in\Lambda} Y_\lambda, p')$ be the product space of a family $\{(X_\lambda, q_\lambda) | \lambda \in \Lambda\}$ (resp. $\{(Y_\lambda, p_\lambda) | \lambda \in \Lambda\}$ of spaces. For each $\lambda \in \Lambda$, $f_\lambda:(X_\lambda, q_\lambda) \to (Y_\lambda, p_\lambda)$ is a functon, $f:(\prod_{\lambda\in\Lambda} X_\lambda, q) \to (\prod_{\lambda\in\Lambda} Y_\lambda, p')$ is the function defined by $f(x)=(f_\lambda(x_\lambda))_{\lambda\in\Lambda}$ for each $x=(x_\lambda)_{\lambda\in\Lambda} \in \prod_{\lambda\in\Lambda} X_\lambda$

By Theorem 2.9, the following corollary is easily obtained.

Corollary 2.11. *f is strongly θ-continuous if and only if $Q_\lambda \cdot f$ is strongly θ-continuous for each $\lambda \in \Lambda$, where $Q_\lambda : \prod_{\lambda\in\Lambda} Y_\lambda \to Y_\lambda$ is the canonical projection*

Lemma 2.12. $\Gamma_{q'}(\prod_{\lambda\in\Lambda} F_\lambda) = \prod_{\lambda\in\Lambda} \Gamma_{q_\lambda}(F_\lambda)$

Proof. Let $x=(x_\lambda)_{\lambda\in\Lambda} \in \prod_{\lambda\in\Lambda} \Gamma_{q_\lambda}(F_\lambda)$, then $x_\lambda \in \Gamma_{q_\lambda}(F_\lambda)$ for each $\lambda \in \Lambda$. There exists an ultrafilter ψ_λ on X_λ such that ψ_λ q_λ-converges to x_λ and $F_\lambda \in \psi_\lambda$ Since $\prod_{\lambda\in\Lambda} \psi_\lambda$ is an ultrafilter containing $\prod_{\lambda\in\Lambda} F_\lambda$ and $\prod_{\lambda\in\Lambda} \psi_\lambda$ q'-converges to $x=(x_\lambda)_{\lambda\in\Lambda}$, $x \in \Gamma_{q'}(\prod_{\lambda\in\Lambda} F_\lambda)$.

Conversely, let $x=(x_\lambda)_{\lambda\in\Lambda} \in \Gamma_{q'}(\prod_{\lambda\in\Lambda} F_\lambda)$, then there exists an ultrafilter ψ on $\prod_{\lambda\in\Lambda} X_\lambda$ such that ψ q'-converges to x and $\prod_{\lambda\in\Lambda} F_\lambda \in \psi$ For each $\lambda \in \Lambda$, $P_\lambda(\psi)$ is an ultrafilter, $P_\lambda(\psi)$ q_λ-converges to x_λ, and $P_\lambda(\prod_{\lambda\in\Lambda} F_\lambda) = F_\lambda \in P_\lambda(\psi)$. Therefore $x_\lambda \in \Gamma_{q_\lambda}(F_\lambda)$.

Theorem 2.13. *f_λ is strongly θ-continuous if and only if $f_\lambda \cdot P_\lambda$ is strongly θ-continuous for each $\lambda \in \Lambda$.*

Proof. Let ψ q'-converge to $x=(x_\lambda)_{\lambda\in\Lambda}$, then $P_\lambda(\psi)$ q_λ-converges to x_λ for each $\lambda \in \Lambda$. Since f_λ is strongly θ-continuous and P_λ is contionouous,

$$V_{p_\lambda}(f_\lambda(P_\lambda(x)))=V_{p_\lambda}(f_\lambda(x_\lambda))\subset f_\lambda(\Gamma_{q_\lambda}(P_\lambda(\psi)))$$
$$\subset f_\lambda(P_\lambda(\Gamma_{q'}(\psi)))$$

Thus $f_\lambda \cdot P_\lambda$ is strongly θ-continuous.

Conversely, let ψ_λ q_λ-converge to x_λ for each $\lambda \in \Lambda$, then $\prod_{\lambda\in\Lambda}\psi_\lambda$ q'- converges to $x=(x_\lambda)_{\lambda\in\Lambda}$. By Lemma 2.12,

$$V_{p_\lambda}(f_\lambda(x_\lambda))=V_{p_\lambda}(f_\lambda(P_\lambda(x)))\subset f_\lambda(P_\lambda(\Gamma_{q'}(\prod_{\lambda\in\Lambda}\psi_\lambda)))$$
$$=f_\lambda(P_\lambda(\prod_{\lambda\in\Lambda}\Gamma_{q_\lambda}(\psi_\lambda)))=f_\lambda(\Gamma_{q_\lambda}(\psi_\lambda)).$$

Thus f_λ is strongly θ-continuous.

Corollary 2.14. *f is strongly θ-continuous if and only if f_λ is strongly θ-continuous for each $\lambda \in \Lambda$.*

Theorem 2.15. *Let $f:(S,q)\to(T,p)$ be continuous If (S,q) is a regular space, then f is strongly θ-continuous.*

A space (S,q) is said to be *semi-regular* if $I_q(\Gamma_q(\psi))$ q- converges to s whenever ψ q-converges to s. Clearly, every regular space is semiregular.

A function f: $(S,q)\to(T,p)$ is said to be *almost strongly θ-continuous* at a point $s\in S$ if $I_p(\Gamma_p(V_p(f(s))))\subset f(\Gamma_q(\psi))$ whenever ψ q-converges to s. f is called *almost strongly θ-continuous* if f is almost strongly θ-continuous at each $s\in S$.

Almost strongly θ-continuous functions are always θ-continuous.

The following theorem is easily consequence of the above definition.

Theorem 2.16. *Let (T,p) be a pretopological semi-regular space. If $f:(S,q)\to(T,p)$ is almost strongly θ-continuous, then f is strongly θ-continuous.*

3. δ-continuous functions and the others functions

A function f from a space (S,q) onto another space (T,p) is said to be *δ-continuous* (resp. *almost continuous, weakly continuous*) at $s\in S$ if $I_p(\Gamma_p(V_p(f(s))))\subset f(I_q(\Gamma_q(\psi)))$ (resp. $I_p(\Gamma_p(V_p(f(s))))\subset f(\psi)$, $\Gamma_p(V_p(f(s)))\subset f(\psi)$) whenever ψ q-converges to s. f is called *δ-continuous* (resp. *almost continuous, weakly continuous*) if f is δ-continuous (resp. almost continuous, weakly continuous) at each $s\in S$.

By above definitions, the following statements are easily obtained.

Theorem 3.1. (1) *Almost strongly θ-continuous functions are always δ-continuous and almost continuous*

(2) *Almost continuous functions are always weakly continuous*

(3) *θ-continuous implies weakly continuous.*

Theorem 3.2. *Let $f:(S,q)\to(T,q)$ and $g:(T,p)\to(R,r)$ be two δ-continuous If (T,p) is a pretopological semi-regular space, then $g\cdot f$ is δ-continuous.*

Proof Let ψ q-converge to s, then $I_p(\Gamma_p(V_p(f(s))))\subset f(I_q(\Gamma_q(\psi)))$. Since (T,p) is a pretopological semi-regular space, $I_p(\Gamma_p(V_p(f(s))))$ p-converges to $f(s)$. Thus

$$I_r(\Gamma_r(V_r(g(f(s)))))\subset g(I_p(\Gamma_p(V_p(f(s)))))$$
$$\subset g(f(I_q(\Gamma_q(\psi))))$$

Theorem 3.3. *Let (T,p) be a pretopological space If $f:(S,q)\to(T,p)$ is strongly θ-continuous and $g:(T,p)\to(R,r)$ is almost continuous, then $g\cdot f$ is almost strongly θ-continuous*

Proof. Let ψ q-converge to s. Since (T,p) is a pretopological space and f is strongly θ-continuous,

$$I_r(\Gamma_r(V_r(g(f(s)))))\subset g(V_p(f(s)))\subset g(f(\Gamma_q(\psi)))$$

by almost continuity of g Thus $g \cdot f$ is almost strongly θ-continuous.

Theorem 3.4. *Let (S,q) be a semi-regular space. If $f:(S,q)\to(T,p)$ is almost continuous, then f is δ-continuous.*

Proof. Let ψ q-converge to s, then $I_q(\Gamma_q(\psi))$ q-converges to s.

A function $f:(S,q)\to(T,p)$ is said to be *super continuous* at $s\in S$ if $V_p(f(s))\subset f(I_q(\Gamma_q(\psi)))$ whenever ψ q-converges to s. f is called *super continuous* if f is super continuous at each $s\in S$.

The following properties are easily obtained by definitions.

Theorem 3.5. (1) *Let (T,p) be a pretopological space. If $f:(S,q)\to(T,p)$ is super continuous and $g:(T,p)\to(R,r)$ is continuous, then $g\cdot f$ is super continuous*

(2) *If $f:(S,q)\to(\prod_{\lambda\in\Lambda} X_\lambda,q')$ is super continuous, then $P_\lambda\cdot f$ is also super continuous for each $\lambda\in\Lambda$.*

(3) *Let $f:(S,q)\to(T,p)$ be a function and $g.(S,q)\to(S\times T,r)$ given by $g(s)=(s,f(s))$, be its graph. If g is super continuous, then f is super continuous.*

Theorem 3.6. *If $f:(S,q)\to(T,p)$ is δ-cntinuous and (T,p) is a pretopological semi-regular space, then f is super continuous*

Theorem 3.7. *Let (T,p) be a pretopological regular space and $f:(S,q)\to(T,p)$ be a function.*

(1) If f is θ-continuous, then f is strongly θ-continuous.

(2) If f is weakly continuous, then f is continuous

(3) If f is almost continuous, then f is continuous.

Since a regular space implies a semi-regular space, by Theorem 2.8, 2.16, 3.1, and 3.7, the following properties are easily verified.

Theorem 3.8. *Let $f:(S,q)\to(T,p)$ be a function and (T,p) be a pretopological regular space Then*

(1) If f is almost strongly θ-continuous, then f is continuous.

(2) Further, the following statements are equivalent ,

(a) f is continuous.

(b) *f is θ-continuous.*

(c) *f is strongly θ-continuous*

(d) *f is weakly continuous.*

References

1. C.W.Baker, *On super continuous functions,* Bull. Korean Math. Soc.,22, No. 1(1985), 17-22.
2. C.H Cook and H.R.Fischer, *Regular convergence spaces,* Math.Ann., 174(1967), 1-7.
3. T Husain, *Topology and maps,* Plenum Press, New York, 1977.
4. D.C Kent, *Convergence functions and their related topologies,* Fund.Math., 54 (1964), 125-133.
5. D C.Kent, *Convergence quotient maps,* Fund.Math., 65(1969), 197-205
6. D.C Kent and G.D.Richardson, *Minimal convergecne spaces,* Trans Amer.Math. Soc., 160(1971), 487-499.
7. D.C.Kent and G.D.Richardson, *Open and proper maps between convergence spaces,* Czech.Math.J., 23(98) (1973), 15-23.
8. N.Levine, *A decomposition of continuity in topological spaces,* Amer.Math.Monthly, 68(1961), 44-46.
9. P.E.Long and L.L.Herrington, *Strongly θ-continuous functions,* J.Korean Math. Soc., 18, No. 1(1981), 21-28.
10. T.Noiri, *Between continuity and weakly-continuity,* Bull.Un.Mat.Ital., (4),9 (1974), 647-654.
11. T.Noiri, *On δ-continuous functions,* J.Korean Math.Soc., 16, No.2 (1980), 161-166.
12. F.J.Papp, *Almost-continuous implies θ-continuous,* Notices Amer.Math.Soc., 20 (1973), A-175
13. I.L.Reilly and M.K.Vamanamurthy, *On super-continuous Mappings,* Indian J. pure appl.Math., 14(6) (1983), 767-772.
14. M.K.Singal and A.R.Singal, *Almost-continuous mappings,* Yokohama Math.J., 16 (1968), 63-73.
15. S.Willard, *General topology,* Addison-Wesley Pub.Co.,Inc., Reading,Mass., 1970.

Dong-A University
Pusan 604-714, Korea