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ON CERTAIN CLASS OF ANALYTIC FUNCTIONS WITH NEGATIVE COEFFICIENTS

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1. Introduction

Let A denote the class of functions of the form

$$f(z)=+\sum_{n=2}^{\infty}a_n z^n \qquad (1.1)$$

which are analytic in the unit disk $U=\{z \cdot |z|<1\}$
And let S denote the subclass of A consisting of analytic and univalent functions $f(z)$ in the unit disk U

A function $f(z)$ in S is said to be starlike of order α if

$$Re\{\frac{zf'(z)}{f(z)}\}>\alpha \ (z\in U) \qquad (1.2)$$

for some $\alpha(0\leq\alpha<1)$ We denote by $S^*(\alpha)$ the class of all starlike functions of order α. Furthermore, a function $f(z)$ in S is said to be convex of order α if

$$Re\{1+\frac{zf''(z)}{f'(z)}\}>\alpha \ (z\in U) \qquad (1.3)$$

for some $\alpha(0\leq\alpha<1)$ And we denote by $K(\alpha)$ the class of all convex functions of order α.

It is well known that $f(z)\in K(\alpha)$ if and only if $zf'(z)\in S^*(\alpha)$, and that $S^*(\alpha)\subseteq S^*(o)\equiv S^*$, and $K(\alpha)\subseteq K(o)\equiv K$ for $0 \leq\alpha<1$

These classes $S^*(\alpha)$ and $K(\alpha)$ were first introduced by Robertson ([6]), and later were studied by MacGreger([4]) and Dinchuk([5]).

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Let $(f*g)(z)$ be the convolution or Hadamard product of two functions $f(z)$ and $g(z)$, that is, if $f(z)$ is given by (1.1) and $g(z)$ is given by

$$g(z)=z+\sum_{n=2}^{\infty}b_n z^n \tag{1.4}$$

Then

$$(f*g)(z)=\sum_{n=2}^{\infty}a_n b_n z^n \tag{1.5}$$

Let T denote the subclass of S consisting of functios $f(z)$ whose nonzero coefficients, from the second on, are negative. That is, an analytic function $f(z)$ is in the class T if it can be expressed as

$$f(z)=z-\sum_{n=2}^{\infty}a_n z^n \quad (a_n\geq 0) \tag{1.6}$$

and let $T^*(\alpha)=T\cap S^*(\alpha)$, $C(\alpha)=T\cap K(\alpha)$.

The class $T^*(\alpha)$ and related classes posses some very interesting properties and have been studied by Silverman([9]) and others. Also, Gupta and Ahmad([2],[3]) introduced the subclasses of T and obtained some of the results of Silverman([9]) for the class $T^*(\alpha)$ and $C(\alpha)$.

For a function $f(z)$ in S, we define

$$D^0 f(z)=f(z) \tag{1.7}$$

$$D^1 f(z)=Df(z)=zf'(z), \text{ and} \tag{1.8}$$

$$D^j f(z)=D(D^{j-1}f(z)) \quad (j=1,2,\cdots). \tag{1.9}$$

The differential operator D^j was introduced by Salagean ([7]).

With the help of the differential operator D^j, we say that a function $f(z)$ belonging to T is said to be in the class $T(j,\alpha)$ if and only if

$$Re\left\{\frac{D^{j+1}f(z)}{D^j f(z)}\right\}>\alpha \quad (j=0,1,2,\cdots) \tag{1.10}$$

for some $\alpha(0\leq\alpha<1)$, and for all z in U.

In particular, the class $T(0,\alpha)$ and $T(1,\alpha)$ was studied by Silverman ([9]).

In the paper, we investigate coefficient estimates and distortion properties for the class $T(j,\alpha)$ Furthermore, we prove that the class $T(j,\alpha)$ is closed under convex linear combinations. Also, we generalize some results of Silverman ([9]) and Schild and Silverman ([8]).

2. Coefficient estimates

Theorem 2.1. Let $f(z)=z+\sum_{n=2}^{\infty}a_n z^n$. If $\sum_{n=2}^{\infty}n^j(n-\alpha)\ a_n < 1-\alpha$, then $Re\{\frac{D^{j+1}f(z)}{D^j f(z)}\}>\alpha(j=0,1,2,\cdots)$

Proof It suffices to show that the values for $\frac{D^{j+1}f(z)}{D^j f(z)}$ lies in a circle centered at $w=1$ whose radius is $1-\alpha$
We have

$$\left|\frac{D^j f(z)}{D^{j+1}f(z)}-1\right|=\left|\frac{D^j f(z)}{D^{j+1}f(z)-D^j f(z)}\right|$$

$$=\left|\frac{\sum_{n=2}^{\infty}(n^{j+1}-n^j)a_n z^n}{z+\sum_{n=2}^{\infty}n^j|a_n||z|^n}\right|$$

$$\leq\frac{\sum_{n=2}^{\infty}n^j(n-1)\ |a_n||z|^{n-1}}{1-\sum_{n=2}^{\infty}n^j\,|a_n||z|^{n-1}}$$

$$\leq\frac{\sum_{n=2}^{\infty}n^j(n-1)\ |a_n|}{1-\sum_{n=2}^{\infty}n^j\,|a_n|}$$

This last expression is bounded above by $1-\alpha$ if

$$\sum_{n=2}^{\infty}n^j(n-1)\,|a_n|\leq\ (1-\alpha)(1-\sum_{n=2}^{\infty}n^j\,|a_n|\,)$$

which is equivalent to

$$\sum_{n=2}^{\infty} n^j(n-\alpha)|a_n| \leq 1-\alpha \tag{2.1}$$

But (2.1) is true by hypothesis. Hence $\left|\frac{D^{j+1}f(z)}{D^jf(z)}-1\right| \leq 1-\alpha$, and the theorem is proved.

The following theorem give a necessary and sufficient condition for a function to be in $T(j,\alpha)$.

Theorem 2.2. A function $f(z)=z-\sum_{n=2}^{\infty}a_nz^n$ is in $T(j,\alpha)$ if and only if

$$\sum_{n=2}^{\infty} n^j(n-\alpha)a_n \leq 1-\alpha. \tag{2.2}$$

Proof In view of theorem 2.1, it suffices to show the only if part. Assume that

$$Re\left\{\frac{D^{j+1}f(z)}{D^jf(z)}\right\}=Re\left\{\frac{z-\sum_{n=2}^{\infty}n^{j+1}a_nz^n}{z-\sum_{n=2}^{\infty}n^ja_nz^n}\right\}>\alpha(|z|<1). \tag{2.3}$$

Choose values of z on the real axis so that $\frac{D^{j+1}f(z)}{D^jf(z)}$ is real. Upon clearing the demominator in (2.3) and letting $z\to 1$ through real values, we obtain

$$1-\sum_{n=2}^{\infty}n^{j+1}a_n \geq \alpha(1-\sum_{n=2}^{\infty}n^ja_n).$$

Thus $\sum_{n=2}^{\infty}n^j(n-\alpha)a_n \leq 1-\alpha$, and the proof is complete. Further, the equality in (2.3) is attained for the function $f(z)$ given by

$$f(z)=z-\frac{1-\alpha}{n^j(n-\alpha)}z^n(z\geq 2). \tag{2.4}$$

Using the Theorem 2.2, we have the following.

Corollary 2.3. $T(j+1,\alpha)\subset T(j,\alpha)$ for $j=0,1,2,\cdots$ and $0\leq\alpha\leq 1$.

Corollary 2.4. Let the function $f(z)$ defined by (1.6) be in the class $T(j,\alpha)$, then

$$a_n\leq\frac{1-\alpha}{n^j(n-\alpha)}\ (n\geq 2) \tag{2.5}$$

3. Distortion Theorems

Using the result obtained in Theorem 2.2, we prove

Theorem 3.1. If $f(z)\in T(j,\alpha)$, then

$$r-\frac{1-\alpha}{2^j(2-\alpha)}r^2\leq|f(z)|\leq r+\frac{1-\alpha}{2^j(2-\alpha)}r^2\quad(|z|=r)$$

with equality for the function $f(z)=z-\dfrac{1-\alpha}{2^j(2-\alpha)}z^2$.

Proof. Note that

$$2^j(2-\alpha)\sum_{n=2}^{\infty}a_n\leq\sum_{n=2}^{\infty}n^j(n-\alpha)a_n\leq 1-\alpha.$$

This last inequality follows from Theorem 2.2.
Thus

$$|f(z)|\leq r+\sum_{n=2}^{\infty}a_n\,r^n\leq r+r^2\sum_{n=2}^{\infty}a_n\leq r+\frac{1-\alpha}{2^j(2-\alpha)}r^2.$$

Similarly,

$$|f(z)|\geq r-\sum_{n=2}^{\infty}a_n\,r^n\geq r-r^2\sum_{n=2}^{\infty}a_n\geq r-\frac{1-\alpha}{2^j(2-\alpha)}r^2.$$

Theorem 3.2. If $f(z)\in T(j,\alpha)$, then

$$1-\frac{1-\alpha}{2^{j-1}(2-\alpha)}r\leq|f'(z)|\leq 1+\frac{1-\alpha}{2^{j-1}(2-\alpha)}r$$

with equality for the function $f(z)=z-\frac{1-\alpha}{2^j(2-\alpha)}z^2$.

Proof. We have

$$|f'(z)|\leq 1+\sum_{n=2}^{\infty}na_n|z|^{n-1}\leq 1+r\sum_{n=2}^{\infty}na_n. \tag{3.1}$$

In view of Theorem 2.2,

$$2^j\sum_{n=2}^{\infty}na_n\leq 1-\alpha+\alpha 2^j\sum_{n=2}^{\infty}a_n$$

$$\leq 1-\alpha+\alpha 2^j\frac{1-\alpha}{2^j(2-\alpha)}=\frac{2(1-\alpha)}{2-\alpha}.$$

A substitution of (3.2) into (3.1) yields the right-hand inequality. On the other hand,

$$|f'(z)|\geq 1-\sum_{n=2}^{\infty}na_n|z|^{n-1}\geq 1-r\sum_{n=2}^{\infty}n\ a_n\geq 1-\frac{1-\alpha}{2^{j-1}(2-\alpha)}r.$$

This completes the proof.

4. Convex Linear Combinations

In this section, we shall prove the class $T(j,\alpha)$ is closed under convex liear combinations.

Theorem 4.1. $T(j,\alpha)$ is a convex set.

Proof. Let the functions

$$f_i(z)=z-\sum_{n=2}^{\infty}a_{i,n}z^n\,(a_{i,}\geq 0,\ i=1,2,) \tag{4.1}$$

be in the class $T(j,\alpha)$. It is sufficient to show that the function $h(z)$ defined by

$$h(z)=\lambda f_1(z)+(1-\lambda)f_2(z)\,(0\leq\lambda\leq 1) \tag{4.2}$$

is in the class $T(j,\alpha)$. Since

$$h(z)=z-\sum_{n=2}^{\infty}\{\lambda a_{1,n}+(1-\lambda)a_{2,n}\}z^n, \tag{4.3}$$

with the aid of Theorem 2.2, we have

$$\sum_{n=2}^{\infty}n^j(n-\alpha)\{\lambda a_{1,n}+(1-\lambda)a_{2,n}\}\leq 1-\alpha \tag{4.4}$$

which implies $h(z)\in T(j,\alpha)$.

Theorem 4.2. Let $f_1(z)=z$ and

$$f_n(z)=z-\frac{1-\alpha}{n^j(n-\alpha)}z^n \quad (j=0,1,2,\cdots,\ n\geq 2) \tag{4.5}$$

for $0\leq\alpha<1$. Then $f(z)$ is in the class $T(j,\alpha)$ if and only if it can be expresed in the form

$$f(z)=\sum_{n=1}^{\infty}\lambda_n f_n(z) \tag{4.6}$$

where $\lambda_n\geq 0$ $(n\geq 1)$ and $\sum_{n=1}^{\infty}\lambda_n=1$.

Proof. Assume that

$$f(z)=\sum_{n=1}^{\infty}\lambda_n f_n(z)=z-\sum_{n=2}^{\infty}\frac{1-\lambda}{n^j(n-\alpha)}\lambda_n z^n. \tag{4.7}$$

Then it follows that

$$\sum_{n=2}^{\infty}\frac{n^j(n-\alpha)(1-\alpha)}{(1-\alpha)n^j(n-\alpha)}\lambda_n$$

$$=\sum_{n=2}^{\infty}\lambda_n=1-\lambda_1\leq 1$$

which shows $f(z)\in T(j,\alpha)$

Conversely, assume that the function $f(z)$ defined by (1.6) belongs to the class $T(j,\alpha)$

Then, since

$$a_n\leq\frac{1-\alpha}{n^j(n-\alpha)}\ (n\geq 2), \tag{4.9}$$

we may put

$$\lambda_n=\frac{n^j(n-\alpha)}{1-\alpha}\,a_n \quad (n\geqq 2), \tag{4.10}$$

and

$$\lambda_1=1-\sum_{n=2}^{\infty}\lambda_n. \tag{4.11}$$

Hence, we can see that $f(z)$ can be expressed in the form (4.6).

Above Theorem 4.2 shows that the class $T(j,\alpha)$ is closed under convex linear combinations.

Corollary 4.3 The extreme points of the class $T(j,\alpha)$ are the functions $f_n(z)$ $(n\geq 1)$ given by Theorem 4.2.

5. Convolution Properties

Let the function $f_i(z)$ $(i=1,2)$ be defined by (4.1). Then, we define the modified convolution $(f_1 * f_2)(z)$ of $f_1(z)$ and $f_2(z)$ by

$$(f_1 * f_2)(z)=z-\sum_{n=2}^{\infty}a_{1,n}a_{2,n}z^n \tag{5.1}$$

Theorem 5.1 Let the function $f_i(z)$ $(i=1,2)$ defined by (4.1) be in the class $T(j,\alpha)$. Then $(f_1 * f_2)(z)$ is in the class $T(j,\beta)$, where

$$\beta=\frac{2^j-2\left(\frac{1-\alpha}{2-\alpha}\right)^2}{2^j-\left(\frac{1-\alpha}{2-\alpha}\right)^2} \tag{5.2}$$

The result is sharp.

Proof. From Theorem 2.2, we know that

$$\sum_{n=2}^{\infty}n^j(n-\alpha)a_n\leq 1-\alpha, \text{ and } \sum_{n=2}^{\infty}n^j(n-\alpha)b_n\leq 1-\alpha$$

We wish to find the largest $\beta=\beta(\alpha)$ such that

$$\sum_{n=2}^{\infty}n^j(n-\beta)a_nb_n\leq 1-\beta.$$

Equivalently, we want to show that

$$\sum_{n=2}^{\infty}\frac{n^j(n-\alpha)}{1-\alpha}a_n\leq 1 \tag{5.3}$$

and

$$\sum_{n=2}^{\infty}\frac{n^j(n-\alpha)}{1-\alpha}b_n\leq 1 \tag{5.4}$$

imply that

$$\sum_{n=2}^{\infty}\frac{n^j(n-\beta)}{1-\beta}a_nb_n\leq 1 \tag{5.5}$$

for all $\beta=\beta(\alpha)\leq\dfrac{n^j-n\left(\dfrac{1-\alpha}{n-\alpha}\right)^2}{n^j-\left(\dfrac{1-\alpha}{n-\alpha}\right)^2}$

From (5.3) and (5.4), we get, by means of the Cauchy-Schwarz inequality,

$$\sum_{n=2}^{\infty}\frac{n^j(n-\alpha)}{1-\alpha}\sqrt{a_nb_n}\leq 1 \tag{5.6}$$

It will be, therefore, sufficient to prove that

$$\frac{n^j(n-\beta)}{1-\beta}a_nb_n\leq\frac{n^j(n-\alpha)}{1-\alpha}\sqrt{a_nb_n}$$

for $\beta\leq\beta(\alpha)$ and $n=2,3,\cdots$

or

$$\sqrt{a_nb_n}\leq\left(\frac{n-\alpha}{1-\alpha}\right)\left(\frac{1-\beta}{n-\beta}\right).$$

From (5.6), it follows that $\sqrt{a_nb_n}\leq\dfrac{1-\alpha}{n^j(n-\alpha)}$ for each n $(n=2,3,\cdots)$.

Hence, it will be sufficient to show that

$$\frac{1-\alpha}{n^j(n-\alpha)}\leq\left(\frac{n-\alpha}{1-\alpha}\right)\left(\frac{1-\beta}{n-\alpha}\right)\text{ for all } n=2,3,\cdots. \tag{5.7}$$

Inequality (5.7) is equivalent to

$$\beta \leq \frac{n^j - n\left(\frac{1-\alpha}{n-\alpha}\right)^2}{n^j - \left(\frac{1-\alpha}{n-\alpha}\right)^2} = \frac{n^j(n-\alpha)^2 - n(1-\alpha)^2}{n^j(n-\alpha)^2 - (1-\alpha)^2} \tag{5.8}$$

The right hand side of (5.8) is a increasing function of n $(n=2,3,\cdots)$. Therefore, setting $n=2$ in (5.8), we get

$$\beta \leq \frac{2^j - 2\left(\frac{1-\alpha}{2-\alpha}\right)^2}{2^j - \left(\frac{1-\alpha}{2-\alpha}\right)^2}$$

The result is sharp, for the functions

$$f_j(z) = z - \frac{1-\alpha}{2^j(2-\alpha)} z^2 \in T(j,\alpha) \quad (j=1,2) \tag{5.9}$$

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