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ON CERTAIN CLASS OF ANALYTIC FUNCTIONS DEFINED BY CONVOLUTIONS

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Abstract

We introduce a class $L^*_\sigma(\alpha, \beta, \gamma)$ of functions defined by $f*S_\sigma(z)$ of $f(z)$ and $S_\sigma(z)=z/(1-z)^{2(1-\sigma)}$. The present paper is to determine extreme point, coefficient inequalities, distortion Theorem and radius of starlikeness and convexity for functions in $L^*_\sigma(\alpha, \beta, \gamma)$. And we give fractional calculus.

1. Introduction

Let A denote the class of functions of the form

$$f(z)=z+\sum_{n=2}^{\infty}a_n z^n \tag{1.1}$$

which are analytic in the unit disk $U=\{z : |z|<1\}$. And let S denote the subclass of A consisting of analytic and univalent functions $f(z)$ in U.

A function $f(z)$ in S is said to be starlike of order α if

$$Re\{\frac{zf'(z)}{f(z)}\}>\alpha(z\in U), \tag{1.2}$$

for some $(0\leqq\alpha<1)$. Denote by $S^*(\alpha)$. Further, a function $f(z)$ in S is said to be convex of order α if,

$$Re\{1+\frac{zf''(z)}{f'(z)}\}>\alpha(z\in U), \tag{1.3}$$

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for some $(0\leqq\alpha<1)$. Denote by $K(\alpha)$. Now, the function

$$S_\delta(z)=\frac{z}{(1-z)^{2(1-\delta)}},(0\leqq\delta<1) \tag{1.4}$$

is the well-known extremal function for the class $S^*(\delta)$. Setting

$$C(\delta,n)=\frac{(k-2\delta)}{(n-1)!}\ (n=2,3,4,\cdots\cdots), \tag{1.5}$$

$S_\delta(z)$ can be written in the form

$$S_\delta(z)=z+\sum_{n=2}^{\infty}C(\delta,n)z^n \tag{1.6}$$

Then we note that $C(\delta,n)$ is decreasing in δ and satisfies

$$\lim_{n\to\infty}C(\delta,n)=\begin{cases}\infty & (\delta<\frac{1}{2})\\ 0 & (\delta>\frac{1}{2})\\ 1 & (\delta=\frac{1}{2})\end{cases} \tag{1.7}$$

Let $f*g(z)$ be the convolution of two functions $f(z)$ and $g(z)$, that is, if $f(z)$ is given by (1.1) and $g(z)$ is given by $g(z)=z+\sum_{n=2}^{\infty}b_nz^n$ then

$$f*g(z)=z+\sum_{n=2}^{\infty}a_nb_nz^n. \tag{1.8}$$

We say that a function $f(z)$ defined by (1.1) belongs to the class $L_\delta(\alpha,\beta,\gamma)$ if $f(z)$ satisfies the following condition

$$\left|\frac{(f*S_\delta(z))'-1}{\alpha(f*S_\delta(z))'+(1-\gamma)}\right|<\beta \tag{1.9}$$

for $0\leqq\alpha\leqq1$, $0<\beta\leqq1$ and $0\leqq\gamma\leqq1$.

$$f(z)\in T\ \ iff\ \ f(z)=z-\sum_{n=2}^{\infty}a_nz^n\ \ (a_n\geqq0) \tag{1.10}$$

We denote $L^*_\sigma(\alpha, \beta, \gamma)=L_\sigma(\alpha, \beta, \gamma)\cap T$. The class $L_\sigma(\alpha, \beta, \gamma)$ is the generalization of the class $L(\alpha, \beta, \gamma)$ which was defined by S K. Lee [1]. In particular, $L_{\frac{1}{2}}(\alpha, \beta, \gamma)=L(\alpha, \beta, \gamma)$.

2. Coefficient Inequalities

Theorem 2.1. Let the function $f(z)$ be defined (1.10). Then $f(z)$ is in the class $L^*_\sigma(\alpha, \beta, \gamma)$ if and only if

$$\sum_{n=2}^{\infty}(1+\alpha\beta)n\ C(\delta, n)a_n \leqq \beta(\alpha+1-\gamma) \tag{2.1}$$

The result is sharp.

Proof. Let $f(z)$ be in the class $L^*_\sigma(\alpha, \beta, \gamma)$ Then we have

$$\left|\frac{(f*S_\sigma(z))'-1}{\alpha(f*S_\sigma(z))'+(1-\gamma)}\right| \tag{2.2}$$

$$=\left|\frac{-\sum_{n=2}^{\infty} n\ C(\delta, n)a_n z^{n-1}}{(\alpha+1-\gamma)-\alpha\sum_{n=2}^{\infty} n\ C(\delta,n)a_n z^{n-1}}\right| <\beta$$

for all $z\in U$. Since the denominator in (2.2) is positive for small positive values of z and, consequently, for all z $(0<z<1)$, we let $z\to 1^-$ to obtain

$$\sum_{n=2}^{\infty} n\ C(\delta, n)a_n \leqq \beta(\alpha+1-\gamma)-\alpha\beta\sum_{n=2}^{\infty} n\ C(\delta, n)a_n \tag{2.3}$$

which is equivalent to (2.1).

For the converse, let the inequality (2.1) hold. Then we obtain that

$$\left|(f*S(z))'-1\right|-\beta\left|\alpha(f*S(z))'+(1-\gamma)\right|$$

$$=\left|-\sum_{n=2}^{\infty} nC(\delta,n)a_n z^{n-1}\right|-\beta\left|(\alpha+1-\gamma)-\alpha\sum_{n=2}^{\infty} nC(\delta,n)a_n z^{n-1}\right| \tag{2.4}$$

$$\leqq\sum_{n=2}^{\infty}(1+\alpha\beta)\,nC(\delta, n)a_n-\beta(\alpha+1-\gamma) \leqq 0$$

Hence, by the maximum modulus Theorem, we can see that $f(z)$ is in the class $L^*_\delta(\alpha, \beta, \gamma)$.

Finally, the result is sharp, with the extremal function being of the form

$$f(z)=z-\frac{\beta(\alpha+1-\gamma)}{(1+\alpha\beta)nC(\delta, n)}z^n \quad \text{for } n\geqq 2. \tag{2.5}$$

Corollary 2.2. Let the function $f(z)$ defined by (1.10) be in the class $L^*_\delta(\alpha, \beta, \gamma)$. Then

$$a_n \leqq \frac{\beta(\alpha+1-\gamma)}{(1+\alpha\beta)nC(\delta,n)} \quad \text{for } n\geqq 2. \tag{2.6}$$

The equality is attained by the function $f(z)$ in (2.5)

Theorem 2.3. Let

$$f_1(z)=z \quad \text{and} \tag{2.7}$$

$$f_n(z)=z-\frac{\beta(\alpha+1-\gamma)}{(1+\alpha\beta)nC(\delta, n)}z^n \quad (n\geqq 2). \tag{2.8}$$

Then $f(z)$ is in the class $L^*_\delta(\alpha, \beta, \gamma)$ if and only if it can be expressed in the form

$$f(z)=\sum_{n=1}^{\infty}\lambda_n f_n(z), \tag{2.9}$$

where $\lambda_n\geqq 0$ for $n=1, 2, 3, \cdots$ and

$$\sum_{n=1}^{\infty}\lambda_n=1. \tag{2.10}$$

Proof. Assume that

$$f(z)=\sum_{n=1}^{\infty}\lambda_n f_n(z)=z-\sum_{n=2}^{\infty}\frac{\beta(\alpha+1-\gamma)}{(1+\alpha\beta)nC(\delta, n)}\lambda_n z^n \tag{2.11}$$
$$=z-\sum_{n=2}^{\infty}a_n z^n$$

where

$$a_n = \frac{(\alpha+1-\gamma)}{(1+\alpha\beta)nC(\delta,\ n)}\lambda_n$$

Then we observe that

$$\sum_{n=2}^{\infty}(1+\alpha\beta)nC(\delta,\ n)\frac{(\alpha+1-\gamma)}{(1+\alpha\beta)nC(\delta,\ n)}\lambda_n \tag{2.13}$$

$$=\beta(\alpha+1-\gamma)\sum_{n=2}^{\infty}\lambda_n$$

$$=\beta(\alpha+1-\gamma)(1-\lambda_1) \leqq \beta(\alpha+1-\gamma)$$

This shows that $f(z) \in L^*_\sigma(\alpha,\ \beta,\ \gamma)$ with the aid of Theorem 2.1.

Conversely, assume that $f(z)$ is in the class $L^*_\sigma(\alpha,\ \beta,\ \gamma)$, remembering the formula

$$\sum_{n=2}^{\infty}\frac{(1+\alpha\beta)nC(\delta,\ n)}{\beta(\alpha+1-\gamma)}a_n \leqq 1,$$

from Theorem 2.1. We may set

$$\lambda_n = \frac{(1+\alpha\beta)nC(\delta,\ n)}{(\alpha+1-\gamma)}a_n \quad (n \geqq 2) \tag{2.14}$$

and we have from (2.10), that is,

$$\sum_{n=2}^{\infty}\lambda_n \leqq 1.$$

Setting

$$\lambda_1 = 1-\sum_{n=2}^{\infty}\lambda_n, \tag{2.15}$$

we have the representation (2.9). Thus we have theorem.

3. Distortion Theorems

Theorem 3.1. If the function $f(z)$ defined by (1.10) is in the $L_\delta^*(\alpha, \beta, \gamma)$, and either $0 \leqq \delta \leqq \frac{5}{6}$ or $|z| \leqq \frac{3}{4}$, then

$$|f(z)| \geqq \max\left\{0, |z| - \frac{\beta(\alpha+1-\gamma)}{4(1+\alpha\beta)(1-\delta)}|z|^2\right\}, \tag{3.1}$$

$$|f(z)| \leqq |z| + \frac{\beta(\alpha+1-\gamma)}{4(1+\alpha\beta)(1-\alpha)}|z|^2. \tag{3.2}$$

The bounds are sharp.

Proof. By virture of Theorem 2.1, we note that

$$|f(z)| \geqq \max\left\{0, |z| - \max_{n\in N-\{1\}} \frac{\beta(\alpha+1-\gamma)}{(1+\alpha\beta)nC(\delta, n)}|z|^n\right\}, \tag{3.3}$$

$$|f(z)| \leqq |z| + \max_{n\in N-\{1\}} \frac{\beta(\alpha+1-\gamma)}{(1+\alpha\beta)nC(\delta, n)}|z|^n \tag{3.4}$$

for $z \in U$. Hence it suffices to deduce that

$$G(\delta, \alpha, \beta, \gamma, |z|, n) = \frac{\beta(\alpha+1-\gamma)}{(1+\alpha\beta)nC(\delta, n)}|z|^n \tag{3.5}$$

is a decresing function of n $(n \geqq 2)$. Since

$$C(\delta, n+1) = \frac{n+1-2\delta}{n} C(\delta, n), \tag{3.6}$$

we can see that, for $|z| \neq 0$,

$$G(\delta, \alpha, \beta, \gamma, |z|, n) \geqq G(\delta, \alpha, \beta, \gamma, |z|, n+1) \textit{ if and only if} \tag{3.7}$$

$$H(\delta, |z|, n) = (n+1)(n+1-2\delta) + n^2|z| \geqq 0 \tag{3.8}$$

It is easy to see that $H(\delta, |z|, n)$ is decresing function of δ for fixed $|z|$. Consequently it follows that

$$H(\delta, |z|, n) \geqq H(5/6, |z|, n) = n^2(1-z) + \tfrac{1}{3}(n-2) \geqq 0. \tag{3.9}$$

for $0 \leqq \delta \leqq 5/6$, $z \in \mathrm{U}$ and $n \geqq 2$.

Further, since $H(\delta, |z|, n)$ is decresing in $|z|$ and increasing in n, we obtain that

$$H(\delta, |z|, n) > H(1, |z|, n) \geqq H(1, 3/4, 2) = 0 \quad (3.10)$$

for $0 \leqq \delta \leqq 1$, $|z| < 3/4$ and $n \geqq 2$. Thus $\max_{n \in N-\{1\}} G(\delta, \alpha, \beta, \gamma, |z|, n)$

is attained at $n=2$.

Finally, since the functions $f_n(z)$ $(n \geqq 2)$ defined in Theorem 2.1 are the extreme points of the class $L^*_\sigma(\alpha, \beta, \gamma)$, we can see that the bounds of the theorem is attained by the function $f_2(z)$, that is,

$$f_2(z) = z - \frac{\beta(\alpha+1-\gamma)}{4(1+\alpha\beta)(1-\delta)} z^2 \quad (3.11)$$

Theorem 3.2. If the function $f(z)$ defined by (1.10) is in the class $L^*_\sigma(\alpha, \beta, \gamma)$ and either $0 \leqq \delta \leqq 1/2$ or $|z| \leqq 1/2$, then

$$1 - \frac{\beta(\alpha+1-\gamma)}{2(1+\alpha\beta)(1-\delta)} |z| \leqq |f'(z)| \leqq 1 + \frac{\beta(\alpha+1-\gamma)}{2(1+\alpha\beta)(1-\delta)} |z|. \quad (3.12)$$

The bounds are sharp.

Proof. It is similar to Theorem 3.1.

4. Some results of convolution

Theorem 4.1. $L^*_\sigma(\alpha, \beta, \gamma)$ is subclass of S if and only if $0 \leqq \delta \leqq 1/2$

Proof Note that the function $f(z)$ defined by (1.10) is in the class S if

$$\sum_{n=2}^{\infty} n|a_n| \leqq 1, ([6]). \quad (4.1)$$

Hence it suffices to prove that

$$(1+\alpha\beta)C(\delta, n) \geqq \beta(\alpha+1-\gamma) \quad (4.2)$$

for the $0 \leqq \delta \leqq \frac{1}{2}$ and $n \geqq 2$ by means of Theorem 2.1.
Since $C(\delta, n) \geqq C(\frac{1}{2}, n) = 1$ for $0 \leqq \delta \leqq \frac{1}{2}$, we can see that, for $0 \leqq \delta \leqq \frac{1}{2}$,

$$(1+\alpha\beta)C(\delta, n) - \beta(\alpha+1-\gamma) \geqq (1+\alpha\beta) - \beta(\alpha+1-\gamma) \geqq 0. \quad (4.3)$$

Conversely, if we assum $\delta > \frac{1}{2}$, then $\lim_{n\to\infty} C(\delta, n) = 0$.
Taking the function $f_n(z)$ given by (2.8), we have

$$f_n'(z) = 1 - \frac{\beta(\alpha+1-\gamma)}{(1+\alpha\beta)C(\delta, n)} z^{n-1} = 0 \quad (4.4)$$

for $z^{n-1} = \frac{(1+\alpha\beta)C(\delta, n)}{\beta(\alpha+1-\gamma)}$

which is less than one for n sufficiently large. Thus $f_n(z)$ is not univalent for $\delta > \frac{1}{2}$ and $n = n(\alpha)$ sufficiently large.

Theorem 4.2. Let the function $f(z)$ defined by (1.10) be in the class $L_\delta^*(\alpha, \beta, \gamma)$ with $0 \leqq \delta \leqq \frac{1}{2}$, then $f(z)$ is a starlike of order τ $(0 \leqq \tau < 1)$ in the disk $|z| < r_1$, where

$$r_1 = \inf_{n \in N - \{1\}} \left\{ \frac{(1+\alpha\beta)(1-\tau)nC(\delta, n)}{\beta(\alpha+1-\gamma)(n-\tau)} \right\}^{1/(n-1)} \quad (4.5)$$

Proof It is a sufficient to show that the values for $\frac{zf'(z)}{f(z)}$ lie in a circle with center at 1 whose radious is $1-\tau$ for $z < r_1$.

$$\left| \frac{zf'(z)}{f(z)} - 1 \right| \leqq \frac{\sum_{n=2}^{\infty}(n-1)|a_n||z|^{n-1}}{1 - \sum_{n=2}^{\infty}|a_n||z|^{n-1}}. \quad (4.6)$$

Thus $\left| \frac{zf'(z)}{f(z)} - 1 \right| \leqq 1-\tau$ if

$$\sum_{n=2}^{\infty}(n-1)|a_n||z|^{n-1} \leqq (1-\tau)\left\{1 - \sum_{n=2}^{\infty}|a_n||z|^{n-1}\right\}$$

which is equivalent to

$$\left(\frac{n-\tau}{1-\tau}\right)|a_n||z|^{n-1} \leqq 1. \tag{4.7}$$

By virture of Theorem 2.1, we need only to find values of $|z|$ for which the inequality

$$\left(\frac{n-\tau}{1-\tau}\right)|z|^{n-1} \leqq \frac{(1+\alpha\beta)nC(\delta,\ n)}{\beta(\alpha+1-\gamma)} \tag{4.8}$$

Solving (4.8) for $|z|$, we obtain the result.

Theorem 4.3 Let the function $f(z)$ defined by (1.10) be in the class $L^*_\delta(\alpha,\beta,\ \gamma)$ with $0\leqq\delta\leqq\frac{1}{2}$. Then $f(z)$ is convex of order $\tau(0\leqq\tau<1)$ in the disk $|z|<r_2$, where

$$r_2 = \inf_{n\in N-\{1\}} \left\{\frac{(1+\alpha\beta)(1-\tau)C(\delta,\ n)}{\beta(\alpha+1-\gamma)(n-\tau)}\right\}1/(1-1) \tag{4.9}$$

5. Fractional Calculus

We need the definitions of fractional derivatives and fractional integrals which were defined by S. Owa([4]).

Theorem 5.1. Let the function $f(z)$ defined by (1.10) be in the class $L^*_\delta(\alpha,\ \beta,\ \gamma)$ with $0\leqq\delta\leqq\frac{1}{2}$. Then

$$|D_z^{-\lambda} f(z)| \geqq \frac{|z|^{1+\lambda}}{\Gamma(2+\lambda)}\left\{1-\frac{\beta(\alpha+1-\gamma)}{2(2+\lambda)(1+\alpha\beta)(1-\alpha)}\right\} \text{ and} \tag{5.1}$$

$$|D_z^{-\lambda} f(z)| \leqq \frac{|z|^{1+\lambda}}{\Gamma(2+\lambda)}\left\{1+\frac{\beta(\alpha+1-\gamma)}{2(2+\lambda)(1+\alpha\beta)(1-\alpha)}\right\} \tag{5.5}$$

for $\lambda>0$ and $z\in U$ The bounds are sharp.

Proof. It is easily known that

$$D_z^{-\lambda} f(z) = \frac{1}{\Gamma(2+\lambda)}\ z^\lambda \left\{z-\sum_{n=2}^{\infty}\frac{\Gamma(n+1)\Gamma(2+\lambda)}{\Gamma(n+1+\gamma)}a_n z^n\right\}$$

Now, we consider the function

$$F(z)=z-\sum_{n=2}^{\infty}\frac{\Gamma(n+1)\Gamma(2+\lambda)}{\Gamma(n+1+\lambda)}a_n z^n \quad \text{for } \lambda>0. \tag{5.6}$$

We not that

$$0<\frac{\Gamma(n+1)\Gamma(2+\lambda)}{\Gamma(n+1+\lambda)}\leqq\frac{2}{2+\lambda} \tag{5.7}$$

for $\lambda>0$ and $n\geqq 2$, and $C(\delta,\ n+1)\geqq C(\delta,\ n)$ for $0\leqq\delta\leqq 1/2$ and $n\geqq 2$. Since $f(z)\in L^*_\sigma(\alpha,\ \beta,\ \gamma)$, by using Theorem 2.1, we obtain

$$\begin{aligned}|F(z)|&\geqq|z|-|z|^2\sum_{n=2}^{\infty}\frac{\Gamma(n+1)\Gamma(2+\lambda)}{\Gamma(n+1+\lambda)}a_n \qquad (5.8)\\ &\geqq|z|-\left(\frac{2}{2+\lambda}\right)|z|^2\sum_{n=2}^{\infty}a_n\\ &\geqq|z|-\frac{\beta(\alpha+1-\gamma)}{2(2+\lambda)(1+\alpha\beta)(1-\delta)}|z|^2,\end{aligned}$$

$$\begin{aligned}|F(z)|&\leqq|z|+|z|^2\sum_{n=2}^{\infty}\frac{\Gamma(n+1)\Gamma(2+\lambda)}{\Gamma(n+1+\lambda)}a_n \qquad (5.9)\\ &\leqq|z|+\left(\frac{2}{2+\lambda}\right)|z|^2\sum_{n=2}^{\infty}a_n\\ &\leqq|z|+\frac{\beta(\alpha+1-\gamma)}{2(2+\lambda)(1+\alpha\beta)(1-\delta)}|z|^2\end{aligned}$$

which gives (5.5)
Further, taking the function $f(z)$ defined by

$$f(z)=z-\frac{\beta(\alpha+1-\lambda)}{4(1+\alpha\beta)(1-\delta)}z^2,$$

we can see that the bounds of the theorem are sharp.

Theorem 5.2. Let the function $f(z)$ defined by (1.10) be in the class $L^*_\delta(\alpha, \beta, \gamma)$ with $0 \leqq \delta \leqq \frac{1}{2}$. Then

$$|D_z^\lambda f(z)| \geqq \frac{|z|^{1-\lambda}}{\Gamma(2-\lambda)} \left\{1 - \frac{\beta(\alpha+1+\gamma)}{2(2-\lambda)(1+\alpha\beta)(1-\delta)} |z|\right\} \quad (5.10)$$

and

$$|D_z^\lambda f(z)| \leqq \frac{|z|^{1-\lambda}}{\Gamma(2-\lambda)} \left\{1 + \frac{\beta(\alpha+1-\gamma)}{2(2-\lambda)(1+\alpha\beta)(1-\delta)} |z|\right\}. \quad (5.11)$$

The bounds are sharp.

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