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ON CERTAIN SUBCLASS OF STARLIKE FUNCTIONS OF ORDER α AND TYPE β

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Abstract.

Let $S^{*}_{0}(\alpha, \beta, \mu)$ denote the class of functions $f(z)=a_1z-\sum_{n=2}^{\infty} a_n z^n$ analytic in the unit disc $U=\{z \; |z|<1\}$ and satisfying the condition

$$\left| \frac{\dfrac{zf(z)}{f(z)}-1}{(1+\mu)\;\beta\left(\dfrac{zf'(z)}{f(z)}-\alpha\right)-\left(\dfrac{zf'(z)}{f(z)}-1\right)} \right| < 1$$

for some $\alpha(0\leq\alpha<1)$, $\beta(0<\beta\leq1)$, $\mu(0\leq\mu\leq1)$ and for all $z\in U$ And it is the purpose of this paper to show a necessary and sufficient condition for the class $S^{*}_{0}(\alpha, \beta, \mu)$, some results for the Hadamard products of two functions $f(z)$ and $g(z)$ in the class $S^{*}_{0}(\alpha, \beta, \mu)$, the distortion theorem and the distortion theorems for the fractional calculus.

1. Introduction

Let A denote the class of functions $f(z)=z+\sum_{n=2}^{\infty} a_n z^n$ analytic in the unit disc $U=\{z \; |z|<1\}$.

A function $f(z)\in A$ is said to be starlike of order $\alpha(0\leq\alpha<1)$ in the unit disc U if

$$Re\{\frac{zf'(z)}{f(z)}\}>\alpha \tag{1.1}$$

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for $z \in U$. And the above condition(1.1) is equivalent to

$$\left| \frac{\frac{zf'(z)}{f(z)} - 1}{2\left(\frac{zf'(z)}{f(z)} - \alpha\right) - \left(\frac{zf'(z)}{f(z)} - 1\right)} \right| < 1.$$

In 1978, Juneja and Mogra [2] gave the following definition.

Definition 1. Let a function $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ be in the class A. Then $f(z)$ is said to be starlike of order α and type β if the condition

$$\left| \frac{\frac{zf'(z)}{f(z)} - 1}{2\beta\left(\frac{zf'(z)}{f(z)} - \alpha\right) - \left(\frac{zf'(z)}{f(z)} - 1\right)} \right| < 1 \qquad (1.2)$$

is satisfied for some $\alpha (0 \leq \alpha < 1)$, $\beta (0 < \beta \leq 1)$ and all $z \in U$ The class of starlike functions of order α and type β is denoted by $S^*(\alpha, \beta)$.

The aim of the present paper is to introduce a subclass of $S^*(\alpha, \beta)$, which we denote it by $S^*(\alpha, \beta, \mu)$.

Definition 2. Let a function $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ be in the class A. Then $f(z)$ is in the class $S^*(\alpha, \beta, \mu)$ it the condition

$$\left| \frac{\frac{zf'(z)}{f(z)} - 1}{(1+\mu)\beta\left(\frac{zf'(z)}{f(z)} - \alpha\right) - \left(\frac{zf'(z)}{f(z)} - 1\right)} \right| < 1 \qquad (1.3)$$

is satisfied for some $\alpha (0 \leq \alpha < 1)$, $\beta (0 < \beta \leq 1)$, $\mu (0 \leq \mu \leq 1)$ and for all $z \in U$.

Remark The special classes $S^*(\alpha, \beta, 1)$, $S^*(\alpha, \frac{1}{2}, 1)$, $S^*(o, \frac{2\delta - 1}{2\delta}, 1)$ $(\delta > \frac{1}{2})$, $S^*(\frac{1-\gamma}{1+\gamma}, \frac{1+\gamma}{2}, 1)$ $(o < \gamma \leq 1)$ and $S^*(1-\alpha, \frac{1}{2}, 1)$

were studied by Juneja and Mogra [2], McCarty [3], Singh [11], [12], Padmanabhan [8] and Eenigenburg [1], respectively.

Furthermore, let $S^*_0(\alpha, \beta, \mu)$ $(0\leq\alpha<1,\ 0<\beta\leq 1,\ 0\leq\mu\leq 1)$ denote the class of functions

$$f(z)=a_1z-\sum_{n=2}^{\infty} a_nz^n \quad (a_1>0,\ a_n\geq 0) \tag{1.4}$$

analytic in U and satisfying the condition (1.3) for some α, β, μ and for all $z\in U$ We note that $S^*_0(\alpha, \beta, 1)$ is studied by Owa[7].

2. A necessary and sufficient condition for the class $S^*_0(\alpha, \beta, \mu)$.

Theorem 1. Let $0\leq\alpha<1$, $0<\beta\leq\dfrac{1}{1+\mu}$ and $0\leq\mu\leq 1$ Then a function $f(z)$ defined by (1.4) is in the class $S^*_0(\alpha,\beta,\mu)$ if and only if

$$\sum_{n=2}^{\infty}\{[2-(1+\mu)\beta]n-(1+\mu)\alpha\beta\}a_n\leq(1+\mu)\beta(1-\alpha)a_1.$$

Proof. Assume that

$$\sum_{n=2}^{\infty}\{[2-(1+\mu)\beta]n-(1+\mu)\alpha\beta\}a_n\leq(1+\mu)\beta(1-\alpha)a_1$$

and let $|z|=1$ Then we have

$$\begin{aligned}
&\left|zf'(z)-f(z)\right|-\left|(1+\mu)\beta(zf'(z)-\alpha f(z))-zf'(z)+f(z)\right|\\
&=\left|\sum_{n=2}^{\infty}(1-n)a_nz^n\right|-\Big|(1+\mu)\beta(1-\alpha)a_1-[(1+\mu)\beta-1]\sum_{n=2}^{\infty}na_nz^n\\
&\quad-[1-(1+\mu)\alpha\beta]\sum_{n=2}^{\infty}a_nz^n\Big|\\
&\leq\sum_{n=2}^{\infty}(n-1)a_n|z|^n-(1+\mu)\beta(1-\alpha)a_1|z|+[1-(1+\mu)\beta]\sum_{n=2}^{\infty}a_n|z|^n\\
&\quad+[1-(1+\mu)\alpha\beta]\sum_{n=2}^{\infty}a_n|z|^n\\
&\leq\Big[\sum_{n=2}^{\infty}\{[2-(1+\mu)\beta]n-(1+\mu)\alpha\beta\}a_n-(1+\mu)\beta(1-\alpha)a_1\Big]|z|\\
&\leq 0.
\end{aligned}$$

Consequently, by the maximum modulus theorem, the function $f(z)$ belongs to the class $S^*_0(\alpha, \beta, \mu)$.

Conversely, assume that

$$\left| \frac{\frac{zf'(z)}{f(z)} - 1}{(1+\mu)\beta\left(\frac{zf'(z)}{f(z)} - \alpha\right) - \left(\frac{zf'(z)}{f(z)} - 1\right)} \right|$$

$$= \left| \frac{\sum_{n=2}^{\infty}(1-n)a_n z^n}{(1+\mu)\beta(1-\alpha)a_1 z - [1+\mu)\beta - 1] \sum_{n=2}^{\infty} n a_n z^n - [1-(1+\mu)\alpha\beta]\sum_{n=2}^{\infty} a_n z^n} \right| < 1.$$

Since $|Re(z)| \leq |z|$ for any z, we have

$$Re\left\{ \frac{\sum_{n=2}^{\infty}(n-1)a_n z^n}{(1+\mu)\beta(1-\alpha)a_1 z - \sum_{n=2}^{\infty}(n+1-(1+\mu)\beta n - (1+\mu)\alpha\beta)a_n z^n} \right\} < 1. \quad (2.1)$$

Choose values of z on the real axis so that $zf'(z)$ is real. Upon clearing the denominator in (2.1) and letting $z \to 1^-$ through real values,

$$\sum_{n=2}^{\infty}(n-1)a_n \leq (1+\mu)\beta(1-\alpha)a_1 - \sum_{n=2}^{\infty}(n+1-(1+\mu)\beta n - (1+\mu)\alpha\beta)a_n$$

This inequality gives the required condition.

Furthermore, the function

$$f(z) = a_1 z - \frac{(1+\mu)\beta(1-\alpha)a_1}{2[2-(1+\mu)\beta] - (1+\mu)\alpha\beta} z^2$$

is an extremal function for the theorem.

3. The Hadamard products.

Let $f*g(z)$ denote the Hadamard product of two functions

$$f(z) = a_1 z - \sum_{n=2}^{\infty} a_n z^n \qquad (a_1 > 0,\ a_n \geq 0)$$

and

$$g(z)=b_1z-\sum_{n=2}^{\infty}b_nz^n \qquad (b_1>0,\ b_n\geq 0)$$

that is,

$$f*g(z)=a_1b_1\,z-\sum_{n=2}^{\infty}a_nb_nz^n$$

Theorem 2. Let $0\leq\alpha_1<1$, $0\leq\alpha_2<1$, $0<\beta_1\leq\frac{1}{1+\mu}$, $0<\beta_2\leq\frac{1}{1+\mu}$ and $0\leq\mu\leq 1$. Let the functions

$$f(z)=a_1z-\sum_{n=2}^{\infty}a_nz^n \qquad (a_1>0,\ a_n\geq 0),$$

$$g(z)=b_1z-\sum_{n=2}^{\infty}b_nz^n \qquad (b_1>0,\ b_n\geq 0)$$

be in the classes $S^*_0(\alpha_1,\ \beta_1,\ \mu)$ and $S^*_0(\alpha_2,\ \beta_2,\ \mu)$, respectively. Then the Hadamard product $f*g(z)$ belongs to the class $S^*_0(\alpha,\ \beta,\ \mu)$, where $\alpha=Min(\alpha_1,\ \alpha_2)$ and $\beta=Max(\beta_1,\ \beta_2)$

Proof. Since $f(z)\in S^*_0(\alpha_1,\ \beta_1,\ \mu)$ and $g(z)\in S^*_0(\alpha_2,\ \beta_2,\ \mu)$, by using Theorem 1, we have

$$\sum_{n=2}^{\infty}\{[2-(1+\mu)\beta_1]n-(1+\mu)\alpha_1\beta_1\}a_n\leq(1+\mu)\beta_1(1-\alpha_1)a_1$$

and

$$\sum_{n=2}^{\infty}\{[2-(1+\mu)\beta_2]n-(1+\mu)\alpha_2\beta_2\}b_n\leq(1+\mu)\beta_2(1-\alpha_2)b_1$$

Hance,

$$\sum_{n=2}^{\infty}a_n\leq\frac{(1+\mu)\beta_1(1-\alpha_1)a_1}{2[2-(1+\mu)\beta_1]-(1+\mu)\alpha_1\beta_1}<a_1$$

and

$$\sum_{n=2}^{\infty}b_n\leq\frac{(1+\mu)\beta_2(1-\alpha_2)b_1}{2[2-(1+\mu)\beta_2]-(1+\mu)\alpha_2\beta_2}<b_1$$

Therefore, for $\alpha=Min(\alpha_1, \alpha_2)$ and $\beta=Max(\beta_1, \beta_2)$,

$$\sum_{n=2}^{\infty}\{[2-(1+\mu)\beta]n-(1+\mu)\alpha\beta\}a_n b_n$$

$$\leq Max\{b_1\sum_{n=2}^{\infty}\{[2-(1+\mu)\beta]n-(1+\mu)\alpha\beta\}a_n,$$

$$a_1\sum_{n=2}^{\infty}\{[2-(1+\mu)\beta]n-(1+\mu)\alpha\beta\}b_n\}$$

$$\leq(1+\mu)\beta(1-\alpha)a_1b_1.$$

Consequently, the Hadamard product $f*g(z)$ is in the class $S^*_0(\alpha, \beta, \mu)$ by Theorem 1.

Corollary 1. Let $0\leq\alpha<1$, $0<\beta\leq\frac{1}{1+\mu}$ and $0\leq\mu<1$.

Let the functions

$$f(z)=a_1z-\sum_{n=2}^{\infty}a_nz^n \qquad (a_1>0,\ a_n\geq 0)$$

and

$$g(z)=b_1z-\sum_{n=2}^{\infty}b_nz^n \qquad (b_1>0,\ b_n\geq 0)$$

be in the same class $S^*_0(\alpha, \beta, \mu)$. Then the Hadamard product $f*g(z)$ belongs to the class $S^*_0(\alpha, \beta, \mu)$.

4. Distortion theorem.

Theorem 3. Let the function $f(z)$ defined by (1.4) be in the class $S^*_0(\alpha, \beta, \mu)$, where $0\leq\alpha<1$, $0<\beta\leq\frac{1}{1+\mu}$ and $0\leq\mu\leq 1$. Then we have

$$|f(z)|\geq a_1|z|-\frac{(1+\mu)\beta(1-\alpha)a_1}{2[2-(1+\mu)\beta]-(1+\mu)\alpha\beta}|z|^2,$$

$$|f(z)|\leq a_1|z|+\frac{(1+\mu)\beta(1-\alpha)a_1}{2[2-(1+\mu)\beta]-(1+\mu)\alpha\beta}|z|^2,$$

$$|f'(z)| \geq a_1 - \frac{2(1+\mu)\beta(1-\alpha)a_1}{2[2-(1+\mu)\beta]-(1+\mu)\alpha\beta}|z|,$$

and

$$|f'(z)| \leq a_1 + \frac{2(1+\mu)\beta(1-\alpha)a_1}{2[2-(1+\mu)\beta]-(1+\mu)\alpha\beta}|z|,$$

for $z \in U$. The equalities hold for the function

$$f(z) = a_1 z - \frac{(1+\mu)\beta(1-\alpha)a_1}{2[2-(1+\mu)\beta]-(1+\mu)\alpha\beta} z^2.$$

Proof Since $f(z) \in S^*_0(\alpha, \beta, \mu)$, we have

$$\sum_{n=2}^{\infty} a_n \leq \frac{(1+\mu)\beta(1-\alpha)a_1}{2[2-(1+\mu)\beta]-(1+\mu)\alpha\beta} \tag{4.1}$$

by using Theorem 1. Furthermore,

$$\sum_{n=2}^{\infty} \frac{n}{2}\{2[2-(1+\mu)\beta]-(1+\mu)\alpha\beta\}a_n$$

$$\leq \sum_{n=2}^{\infty}\{2[2-(1+\mu)\beta]-(1+\mu)\alpha\beta\}a_n$$

$$\leq (1+\mu)\beta(1-\alpha)a_1, \tag{4.2}$$

that is,

$$\sum_{n=2}^{\infty} n a_n \leq \frac{2(1+\mu)\beta(1-\alpha)a_1}{2[2-(1+\mu)\beta]-(1+\mu)\alpha\beta}$$

Hence,

$$|f(z)| \geq a_1|z| - |z|^2 \sum_{n=2}^{\infty} a_n$$

$$\geq a_1|z| - \frac{(1+\mu)\beta(1-\alpha)a_1}{2[2-(1+\mu)\beta]-(1+\mu)\alpha\beta}|z|^2.$$

$$|f(z)| \leq a_1|z| + |z|^2 \sum_{n=2}^{\infty} a_n$$

$$\le a_1|z| + \frac{(1+\mu)\beta(1-\alpha)a_1}{2[2-(1+\mu)\beta]-(1+\mu)\alpha\beta}|z|^2,$$

$$|f'(z)| \ge a_1 - |z|\sum_{n=2}^{\infty} n a_n$$

$$\ge a_1 - \frac{2(1+\mu)\beta(1-\alpha)a_1}{2[2-(1+\mu)\beta]-(1+\mu)\alpha\beta}|z|$$

and

$$|f'(z)| \le a_1 + |z|\sum_{n=2}^{\infty} n a_n$$

$$\le a_1 + \frac{2(1+\mu)\beta(1-\alpha)a_1}{2[2-(1+\mu)\beta]-(1+\mu)\alpha\beta}|z|$$

for $z \in U$

Corollary 2. Under the hypotheses of Theorem 3, $f(z)$ and $f'(z)$ are included in the discs with center at the origin and radii

$$\frac{[4-(1+\mu)\beta-2(1+\mu)\alpha\beta]a_1}{4-2(1+\mu)\beta-(1+\mu)\alpha\beta} \text{ and } \frac{[4-3(1+\mu)\alpha\beta]a_1}{4-2(1+\mu)\beta-(1+\mu)\alpha\beta},$$

respectively.

5. The distortion theorems for the fractional calculus.

There are many definitions of the fractional calculus. In 1978, Owa [6] gave the following definitions for the fractional calculus.

Definition 3. The fractional integral of order k is defined by

$$D_z^{-k} f(z) = \frac{1}{\Gamma(k)} \int_0^z \frac{f(\xi)d\xi}{(z-\xi)^{1-k}},$$

where $k>0$, $f(z)$ is an analytic function in a simply connected region of the z-plane containing the origin and the multiplicity of $(z-\xi)^{k-1}$ is removed by requiring $\log(z-\xi)$ to be real when $(z-\xi)>0$

Definition 4. The fractional derivative of order k is defined by

$$D_z^k f(z)=\frac{1}{\Gamma(1-k)}\frac{d}{dz}\int_0^z \frac{f(\xi)\,d\xi}{(z-\xi)^k},$$

where $0\leq k<1$, $f(z)$ is an analytic function in a simply connected region of the z-plane containing the origin and the multiplicity of $(z-\xi)^{-k}$ is removed by requiring $\log(z-\xi)$ to be real when $(z-\xi)>0$.

Definition 5. Under the hypotheses of Definition 4, the fractional derivative of order $(n+k)$ is defined by

$$D_z^{n+k} f(z)=\frac{d^n}{dz^n} D_z^k f(z),$$

for $0\leq k<1$ and $n\in N\cup\{0\}$

For other definitions of the fractional calculus, see Nishimoto [4], Osler [5], Ross [9] and Saigo [10].

Theorem 4. Let the function $f(z)$ defined by (1.4) be in the class $S_0^*(\alpha,\beta,\mu)$, $0\leq\alpha<1$, $0<\beta\leq\dfrac{1}{1+\mu}$ and $0\leq\mu\leq 1$. Then we have

$$\left|D_z^{-k} f(z)\right| \geq \frac{|z|^{1+k}a_1}{\Gamma(2+k)}\left\{1-\frac{2(1+\mu)\beta(1-\alpha)}{(2+k)\{2[2-(1+\mu)\beta]-(1+\mu)\alpha\beta\}}|z|\right\}$$

and

$$\left|D_z^{-k} f(z)\right| \leq \frac{|z|^{1+k}a_1}{\Gamma(2+k)}\left\{1+\frac{2(1+\mu)\beta(1-\alpha)}{(2+k)\{2[2-(1+\mu)\beta]-(1+\mu)\alpha\beta\}}|z|\right\}$$

for $0<k<1$ and $z\in U$. The result is sharp.

Proof. Let

$$F(z)=\Gamma(2+k)z^{-k}\, D_z^{-k} f(z)$$
$$=a_1 z-\sum_{n=2}^{\infty}\frac{\Gamma(n+1)\,\Gamma(2+k)}{\Gamma(n+1+k)}\,a_n z^n$$

$$=a_1z-\sum_{n=2}^{\infty}A(n)\ a_nz^n,$$

where

$$A(n)=\frac{\Gamma(n+1)\ \Gamma(2+k)}{\Gamma(n+1+k)}\quad (n\geq 2).$$

Since

$$0<A(n)\leq A(2)=\frac{2}{2+k},$$

we have, with the help of Theorem 1,

$$|F(z)|\geq a_1|z|-A(2)|z|^2\sum_{n=2}^{\infty}a_n$$

$$\geq a_1|z|-\frac{2(1+\mu)\beta(1-\alpha)a_1}{(2+k)\{2[2-(1+\mu)\beta]-(1+\mu)\alpha\beta\}}|z|^2$$

and

$$|F(z)|\leq a_1|z|+A(2)|z|^2\sum_{n=2}^{\infty}a_n$$

$$\leq a_1|z|+\frac{2(1+\mu)\beta(1-\alpha)a_1}{(2+k)\{2[2-(1+\mu)\beta]-(1+\mu)\alpha\beta\}}|z|^2$$

which prove the inequalities of Theorem 4. Further, equalities are attained for the function

$$D_z^{-k}f(z)=\frac{z^{1+k}\ a_1}{\Gamma(2+k)}\left\{1-\frac{2(1+\mu)\beta(1-\alpha)}{(2+k)\{2[2-(1+\mu)\beta]-(1+\mu)\alpha\beta\}}\ z\right\}$$

or

$$f(z)=a_1z-\frac{(1+\mu)\beta(1-\alpha)a_1}{2[2-(1+\mu)\beta]-(1+\mu)\alpha\beta}\ z^2.$$

Thus we complete the proof of Theorem 4.

Corollary 3. Under the conditions of Theorem 4, $D_z^{-k}f(z)$ is included in the disc with center at the origin and radius

$$\frac{a_1}{\Gamma(2+k)}\left\{\frac{2(2+k)[2-(1+\mu)\beta]+2(1+\mu)\beta-(1+\mu)\alpha\beta(4+k)}{(2+k)\{2[2-(1+\mu)\beta]-(1+\mu)\alpha\beta\}}\right\}.$$

Theorem 5. Let the function $f(z)$ defined by (1.4) be in the class $S^*_0(\alpha, \beta, \mu)$, $0\leq\alpha<1$, $0<\beta\leq\frac{1}{1+\mu}$ and $0\leq\mu\leq 1$ Then we have

$$\left|D_z^k f(z)\right| \geq \frac{|z|^{1-k}a_1}{\Gamma(2-k)}\left\{1-\frac{2(1+\mu)\beta(1-\alpha)}{(2-k)\{2[2-(1+\mu)\beta]-(1+\mu)\alpha\beta\}}|z|\right\}$$

and

$$\left|D_z^k f(z)\right| \leq \frac{|z|^{1-k}a_1}{\Gamma(2-k)}\left\{1+\frac{2(1+\mu)\beta(1-\alpha)}{(2-k)\{2[2-(1+\mu)\beta]-(1+\mu)\alpha\beta\}}|z|\right\}$$

for $0\leq k<1$ and $z\in U$. The result is sharp.

Proof Let

$$\begin{aligned} G(z)&=\Gamma(2-k)z^k D_z^k f(z)\\ &=a_1z-\sum_{n=2}^{\infty}\frac{\Gamma(n+1)\Gamma(2-k)}{\Gamma(n+1-k)}a_nz^n\\ &=a_1z-\sum_{n=2}^{\infty}nB(n)a_nz^n, \end{aligned}$$

where

$$B(n)=\frac{\Gamma(n)\,\Gamma(2-k)}{\Gamma(n+1-k)}\quad(n\geqq 2).$$

Noting

$$0<B(n)\leqq B(2)=\frac{1}{2-k},$$

with (4.2), we have

$$\begin{aligned} |G(z)| &\geqq a_1|z|-B(2)|z|^2\sum_{n=2}^{\infty}na_n\\ &\geqq a_1|z|-\frac{2(1+\mu)\beta(1-\alpha)a_1}{(2-k)\{2[2-(1+\mu)\beta]-(1+\mu)\alpha\beta\}}|z|^2 \end{aligned}$$

and

$$|G(z)| \leqq a_1 |z| + B(2) |z|^2 \sum_{n=2}^{\infty} n a_n$$

$$\leqq a_1 |z| + \frac{2(1+\mu)\beta(1-\alpha)a_1}{(2-k)\{2[2-(1+\mu)\beta]-(1+\mu)\alpha\beta\}} |z|^2$$

which give the inequalities of Theorem 5. Since equalities are attained for the function $f(z)$ defined by

$$D_z^k f(z) = \frac{z^{1-k}\, a_1}{\Gamma(2-k)} \{1 - \frac{2(1+\mu)\beta(1-\alpha)}{(2-k)\{2[2-(1+\mu)\beta]-(1+\mu)\alpha\beta\}} z\},$$

that is, by

$$f(z) = a_1 z - \frac{(1+\mu)\beta(1-\alpha)a_1}{2[2-(1+\mu)\beta]-(1+\mu)\alpha\beta} z^2,$$

we complete the assertion of Theorem 5.

Corollary 4. Under the hypotheses of Theorem 5, $D^k{}_z f(z)$ is included in the disc with center at the origin and radius

$$\frac{a_1}{\Gamma(2-k)} \{\frac{2(2-k)[2-(1+\mu)\beta]+2(1+\mu)\beta-(1+\mu)\alpha\beta(4-k)}{(2-k)\{2[2-(1+\mu)\beta]-(1+\mu)\alpha\beta\}}\}$$

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