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V-RINGS DETERMINED BY POLYNOMIAL RINGS

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Let R be a ring with identity. A nonzero left R-module M is called *irreducible* (or simple) if O and M are the only R-submodules.

If R is a division ring, then $R[x]$ is a principal left and right ideal domain. For an irreducible right $R[x]$-module M, $M \simeq R[x]/I$ as a right $R[x]$-module and I is a maximal right ideal of $R[x]$ But since $R[x]$ is a principal ideal domain, there is a monic polynomial $f(x)$ in $R[x]$ such that $I=f(x)R[x]$ Now say $\deg f(x) = n$. Then the irreducible $R[x]$-module $M \simeq R[x]/I$ is generated by $1+I, x+I, \cdots, x^{n-1}+I$ over R as an R-module.

By this well-kown standard fact, it is quite natural to raise the following question at least when R is a simple.

Question 1. Suppose R is a simple ring. Then is every irreducible right $R[x]$-module finitely generated R-module?

But the following example by Resco[7, Example 3.3] nullifies our hope for the affirmative answer for the above question.

Example 2 [Resco]. Let k be a field of characteristic zero and let $K=k[[x]]$ be the ring of power series over k. Then K is a domain. Now let $L=k((z))$ be the field of fractions of K. Let $d \cdot K \to k$ be ordinary differentiation and extended d to L in the usual manner. Let $A= K[y,d]$ the differential operator ring over K. That is, A is free left K-module with basis $\{1, y, y^2, \cdots\}$ and with multiplication extended from K via $ya = ay + d(a)$ for every a in K Also $B=L[y.d]$. Define a $A[x]$-module structure on B; for b in B and $f=\sum x^i a_i$ in $A[x]$, define

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$$b \cdot f = b(\sum x^i a_i) = \sum z^{-i} b a_i$$

Then B is a right $A[x]$-module. Moreover B is an irreducible faithful $A[x]$-module. But B, as a right A-module, is not finitely generated.

But despite of the above pathological example, we are able to give an affirmative answer to our Question 1 in some situaion. Indeed we observe several cases for which every irreducible $R[x]$ -module is a finitely generated R-module.

Definition 3. For a ring R, a right R-module M is called *bounded* if the annihilator $\mathrm{Ann}_R(M)$ of M in R is nonzero.

For example of bounded modules let D be a division ring with the center field F. As we have known that, if D is not purely transcendental over F, then the polynomial ring $D[x]$ over D is not primitive. So in this case every irreducible $D[x]$-module is bounded. For, if it were not, then there would exist an irreducible $D[x]$-module M with $\mathrm{Ann}_{D[x]}(M)=0$. Thus M is a faithful $D[x]$-module and so D[x] is primitive. But this is a contradiction.

The following lemma is a well-known standard fact.

Lemma 4 [Bergman]. Let S be a finite centralizing ring extension of a ring R. If M is an irreducile S-module, then as a R-module, M is a finite direct sum of irreducible R-modules.

The following theorem gives an affirmative answer to Question 1 in some circumstance.

We recall that an overring S of a ring R with same identity is called a *centralizing extension of R with finite basis* if S is finitely generated as an R-module with centralizing finite basis $\{u_1, u_2, \cdots, u_n\}$, that is,

$$S=\sum_{i=1}^{n} r u_i \text{ with } r u_i = u_i r \text{ for each } r \text{ in } R,\ i= 1,2,\cdots,n$$

$$\text{and if } r_1 u_1 + r_2 u_2 + \cdots + r_n u_n = 0, \text{ then } r_1 = r_2 = \cdots = r_n = 0$$

Theorem 5. For a simple ring R any bounded irreducible right

$R[x]$-module is a finite direct sum of irreducible R-module. So it is finitely generated as a right R-module. Moreover, any bounded irreducible right $R[x]$-module never be a projective R-module unless R is right Artinian.

Proof. Let M be a bounded irreducible right $R[x]$-module. Then there is a maximal right ideal I of $R[x]$ such that M is $R[x]$- module isomorophic to $R[x]/I$. Now since M is a bounded $R[x]$- module, $A=\mathrm{Ann}_{R[x]}(M)$ is a nonzero ideal of $R[x]$ contained in I. Pick

$$g(x)=a_0+a_1x+\cdots+a_nx^n$$

a nonzero polynomial in A with the least degree. Then since we may assume $a_n\neq 0$, Ra_nR is a nonzero ideal of R. By our assumption since R is simple, $Ra_nR=R$ so we have

$$1=c_1a_nd_1+c_2a_nd_2+\cdots+c_ka_nd_k$$

for some a_i and d_i in R, $i=1,2,\cdots,k$. Thus

$$f(x)=c_1g(x)d_1+c_2g(x)d_2+\cdots+c_kg(x)d_k$$

is a monic polynomial in A with the least degree. Now for any r in R, $rf(x)-f(x)r=0$, i.e., $rf(x)=f(x)r$ for any r in R. Hence $f(x)$ is a monic central polynomial in $R[x]$. In this case we may use the division algorithm with $f(x)$ and get $A=f(x)R[x]$ since $f(x)$ is monic central. Now if we denote F the center of R, then F is a field and

$$R[x]/A=R\otimes_F F[x]/f[x]F[x]$$

But since F is a field, as a vector space over F, $F[x]/f[x]F[x]$ is generated by $1+f(x)f[x],x+f(x)f[x],\cdots,x^{n-1}+f(x)F[x]$, where n is the degree of the polynomial $f(x)$ Therefore $1+f(x)R[x],x+f(x)R[x],\cdots,x^{n-1}+f(x)R[x]$, i.e., $1+A$, $x+A,\cdots$, $x^{n-1}+A$ is a finite centralizing element of $R[x]/A$ over the ring R. Now since M is an irreducible $R[x]$-module and $A=\mathrm{Ann}_{R[x]}M$, considering M as an $R[x]/A$-module it is also an irreducible $R[x]/A$-module. But since $R[x]/A$ is a finite centralizing extension of R, irreducible right $R[x]/A$-module M is a finite direct sum of irreducible right R-module by Lemma 4. Therefore this $R[x]$-module M is finitely generated as an R-module.

Moreover, if M is a projective R-module considering as a right R-module, then its irreducible R-direct summand is also a projective R-module. So the Socle(R) the sum of the minimal right ideal of R is nonzero because any minimal right ideal of R is R-isomorphic to an irreducible R-direct summand of M. But since R is simple, Soc $(R)=R$. Thus R is simple Artinian. Therefore a bounded irreducible right $R[x]$-module never be a R-projective module unless R is Artinian.

Corollary 6. Let R be a simple ring, Then if the polynomial ring $R[x]$ is not primitive, any irreducible right $R[x]$-module is a finite direct sum of irreducible R-modules.

Proof Since the polynomial ring $R[x]$ is not primitive, any irreducible right $R[x]$-module should be bounded. Therefore it follows immediately by the proof of Theorem 5.

By the hint of Theorem 5 we are able to come to the consideration of ring R such that every irreducible $R[x]$-module is a projective R-module. The forthcoming result may characterize semi-simple Artinian ring in a new way via $R[x]$-module structure. Roughly speaking, the R-projective module property of irreducible $R[x]$-modules influences very strongly on the ring structure of R so that the ring R becomes semisimple Artinian ring.

Theorem 7. The followings are equivalent.

(i) Every irreducible right $R[x]$-module is a projective right R-module.

(ii) R is semisimple Artinian.

Proof (ii) inplies (i). Suppose R is semi-simple Artinian. Then any right R-module is projective. So it is obvious.

(i) implies (ii). Assume that every ireducible $R[x]$-module is a projective right R-module. Define a map θ from $R[x]$ to R by $\theta(f(x))=f(0)$ for $f(x)$ in $R[x]$. Then θ is a ring epimorphism. We *claim* that every irreducible R-module is a projective R-module. Now for an irreducible right R-module R/B with B a maximal right ideal B of R, let

$$I=\{f(x)\in R[x] \mid \theta(f(x))\in B\}.$$

Then since B is a right ideal of R, I is a right ideal fo $R[x]$ and I contains a two-sided ideal $xR[x]$ of $R[x]$.

Pick $f(x) \in R[x]-I$. Then $f(0) \in R-B$ and so $f(0)+B$ is a generator of R/B i.e., $(f(0)+B)R=R/B$. In this case $R[x]/I$ is generated by $f(x)+I$ For, if $g(x)+I$ is in $R[x]/I$, then $g(0)+B=f(0)r+B$ for some r in R. Hence $g(0)-f(0)r$ is in B and therefore $g(x)-f(x)r$ in I. So $g(x)+I=(f(x)+I)f=f(x)r+I$ This means that $R[x]/I$ is an irreducible $R[x]$-module.

Now as R-modules, $R[x]/I$ is isomorphic to R/B. For if we define $\bar{\theta}: R[x]/I \to R/B$ by $\bar{\theta}(f(x)+I)=f(0)+B$, then it can be straightfowardly checked that $\bar{\theta}$ is an R-isomorphism. So $R[x]/I$ is a cyclic R-module. Now by assumption, since $R[x]/I$ is a peojective R-module and so is R/B. Therefore every irreducible R-module is projective. Hence every maximal right ideal is an R-direct summand of R because the exact sequence

$$O \to B \to R \to R/B \to O$$

of R-module with B a maximal right idedal of R is splitted.

Finally to finish our proof, let J be a right ideal of R and let K be a maximal complement of J in R, that is, K is a right ideal maximal with respect to the property $J \cap K=O$. Actually, the existence of maximal complement is assured by Zorn's lemma. Then $J+K=J \oplus K$ is an essential right ideal of R. If $J \oplus K \leqq R$, then there is a maximal right ideal M of R such that $J \oplus K \subseteq M \subsetneqq R$ In this case M is essential since $J \oplus K$ is essential. But by our uesult in the previous paragraph M is a direct summand of R. This is impossible and so $J \oplus K=R$ Therefore every right ideal J of R is an R-direct summand of R. Hence R is semi-simple Artinian ring.

As in the proof of the above theorem a ring whose every irreducible R-module is projective is simple Artinian. With this fact the following definition may be of interest.

Definition 8. A ring R is called a *right V-ring* if every right irreducible R-module is injective.

In the process of the proof for Therem 7, we get following.

Theorem 9. If every irreducible right $R[x]$-module is an injective R-module, then R is a right V–ring.

By this result we may consider its converse.

Lemma 10 [Armendariz and Fisher]. Let R be a P.I.-ring. Then R is a von Neumann ring if and only if R is a V-ring.

Lemma 11 [Posner]. Let R be a prime P.I.-ring. Then R has the classical right quotient ring $Q(R)$ which is simple Artinian. Also in this case $Q(R)$ is the classical left quotient ring and $Q(R)=RF$, where F is the center of the simple Artinian ring $Q(R)$. Moreover the field F is a field of fraction of the domain $Z(R)$.

Lemma 12. Every prime, von Neumann regular P.I.-ring is simple Artinian.

Proof Let R be a prime, von Neumann regular P.I.-ring. Since R is a von Neumann regular ring, then so is $Z(R)$. Now let $0\neq a\in Z(R)$, then there is b in $Z(R)$ such that $a=aba$ But since R is prime, so is $Z(R)$ and hence $Z(R)$ is a commutative domain. From the fact $a=aba$, we have $a(1-ba)=0$ in $Z(R)$ So $1=ba$, since $a\neq 0$. Thus $Z(R)$ is a field. So the center F of $Q(R)$ is $Z(R)$ by Lemma 11. Thus $Q(R)=RF=RZ(R)=R$ and so R is simple Artinian by Lemma 11.

Lemma 13. Let R be a *P.I.*-ring. Then the followings are equivalent.

(i) Every irreducible right $R[x]$-module is finitely generated over R as a module.

(ii) Every primitive factor ring of $R[x]$ is finitely generated over R as a module.

Proof (i) implies (ii) Suppose every irreducible $R[x]$-module is a finitely generated R-module. Let A be a two-sided ideal of $R[x]$ such that $R[x]/A$ is a primitive ring. Then the ring $R[x]/A$ is a primitive P.I.-ring and so it is simple Artinian by Kaplansky. So the ring $R[x]/A$ has a minimal right ideal I_0. In this case I_0 has the form I/A with $A\subseteq I$ and I is a right ideal of $R[x]$. We claim that I/A is an irreducible $R[x]$-module. By the module structure defined by

$$(f(x)+A)g(x)=f(x)g(x)+A$$

for $f(x)+A$ in I/A and $g(x)$ in $R[x]$, I/A is a right $R[x]$-module compatible with the original module structure of I/A as $R[x]/A$-module. By this newly induced module structure on I/A, since I/A is an irreducible $R[x]/A$-module, I/A is an irreducible $R[x]$-module. Finally, since

$$R[x]/A \simeq \oplus \sum I_0$$

as $R[x]$-module and I_0 is finitely generated as R-module by assumption, we have that $R[x]/A$ also is finitely generated as an R-module.

(ii) implies (i) Suppose every primitive factor ring of $R[x]$ is finitely generated as an R-modle. Now let $M=R[x]/N$ be an irreducible $R[x]$-module with N a maximal right ideal of $R[x]$. If $A=\mathrm{Ann}_{R[x]}(M)$, then $R[x]/N$ is a faithful irreducible $R[x]/A$-module. Hence $R[x]/A$ is a primitive P.I.-ring. do $R[x]/A$ is simple Artinian by Kaplansky and hence $A \neq 0$ Now as $R[x]/A$-module we have

$$R[x]/A \simeq \oplus \sum R[x]/N$$

since $R[x]/A$ is simple Artinian and $R[x]/N$ is an irreducible $R[x]/A$-module. So $M=R[x]/N$ is a finitely generated R-module because $R[x]/A$ is a finitely generated R-module.

As a byproduct of the abobe lemma we have the following.

Proposition 14. Let R be a P.I. – ring. Then the followings are equivalent.

(i) Every irreducible $R[x]$-module is a finitetly generated R-module.

(ii) Every maximal ideal of $R[x]$ can be contracted to a maximal ideal of R.

Proof (i) implies (ii) Let A be a maximal ideal of $R[x]$. Then $R/A\cap R \subseteq R[x]/A$ and $R[x]/A$ is simple P.I. So $R[x]/A$ is simple Artinian. By Lemma 13, $R[x]/A$ is a finitely generated R-module. Therefore $R[x]/A$ is a finitely generated $R/A\cap R$-module. Hence $R[x]/A$ is a finite centralizing extension of $R/A\cap R$ Hence $R/A\cap R$ is Artinian, since $R[x]/A$ is Artinian. Now since $R/A\cap R$ is prime P.I., it is simple Artinian. So $A\cap R$ is maximal in R.

(ii) *implies* (i) Let A be a primitive ideal of $R[x]$. Then since $R[x]$ is P.I., A is a maximal ideal. So by our assumption. $A\cap R$ is a maximal ideal of R. Hence $R/A\cap R$ is simple P.I. and so it is simple Artinian. Therefore

$$R[x]/A=\frac{(R/A\cap R)[x]}{A/(A\cap R)[x]}$$

is a finitely generated $R/A\cap R$-module because $R/A\cap R$ is simple Arinian. Hence $R[x]/A$ is a finitely generated R-module. Thus by Lemma 13, we get our conclusion.

Now we are in the situation to characterize V-ring whenever it satisfies a polynomial identity.

Theorem 15. Let R be a P.I.-ring. Then the followings are equivalent.

(i) Every irreducible right $R[x]$-module is an injective R-module.

(ii) R is a (right) V-ring.

Proof. By Theorem 9, (i) implies (ii) immediately. Now suppose R is a (right) V-ring. Then by Lemma 10, R is a von Neumann regular ring. For an irreducible right $R[x]$-module M, let $A=\mathrm{Ann}_{R[x]}(M)$ the annihilator of M in $R[x]$. Then the ring $R[x]/A$ has M as a faithful irreducible module. So $R[x]/A$ is a primitive ring. Our *claim* is that the subring $R+A/A\simeq R/R\cap A$ of $R[x]/A$ is a prime ring. For this, let $\bar{U}$ and $\bar{V}$ be ideals f of $R/R\cap A$ such that $\bar{U}\bar{V}=\bar{0}$. Then there are ideals U,V of R such that $\bar{U}=U/R\cap A$ and $\bar{V}=V/R\cap A$. Then of course $\bar{U}=U+A/A$. So $U[x]$ and $V[x]$ are ideals of $R[x]$, and

$$(U[x]+A)(V[x]+A)\subseteq A.$$

Thus $(U[x]+A/A)(V[x]+A/A)=\bar{0}$ in $R[x]/A$. But since $R[x]/A$ is prime, we have either $U[x]+A/A=\bar{0}$ or $V[x]+A/A=I$ in $R[x]/A$. Therefore $U[x]\subseteq A$ or $V[x]\subseteq A$. Hence $U\subseteq A\cap R$ or $V\subseteq A\cap R$. This means $\bar{U}=\bar{0}$ of $\bar{V}=\bar{0}$. So $R+A/A\simeq R/R\cap A$ is a prime ring.

On the other hand, since R is a von Neumann rigular ring, its homomorphic *image* $R/R\cap A$ *is also a von Neumann regular ring.* Hence the ring. $R/R\cap A$ is a prime, von Neumann regular ring satisfying a polyomial identity. So by Lemma 12, $R/R\cap A$ is simple Artinian. Thus the ideal $R\cap A$ is a maximal ideal.

By this argument so far done in the proof we have that $R \cap A$ is maximal whenever A is a primitive ideal of $R[x]$. But since $R[x]$ is a P.I.-ring, every primitive ideal of $R[x]$ is a maximal ideal. So every maximal ideal of $R[x]$ can be contracted to a maximal ideal of R. But note that in the proof of Proposition 14 every irreducible $R[x]$-module is a *finite direct sum* of an irreducible R-module whenever every maximal ideal of R.

Returning to our situation, M is a finite direct sum of irreducible R-module. Now finally by condition (ii) since irreducible R-module is injective, we have that M is an injective R-module. This completes the proof.

Example 16. Without P.I.-ness of R, Theorem 15 is not true. Let V be an infinite dimensional vector space over a field K. Let S be the socle of $\mathrm{End}_K(V)$, and let $R = S + K1$ Then R is a von Neumann ring but not P.I. In this case as Villamayor and Michler pointed out, as a right R-module, V is irreducible but not injective.

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