On Effects of Large-Deflected Beam Analysis by Iterative Transfer Matrix Approach

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ABSTRACT

A small-deflected beam can be easily solved by assuming a linear system. But a large-deflected beam can not be solved by superposition of the displacements, because the system is nonlinear. The solutions for the large-deflection problems can not be obtained directly from elementary beam theory for linearized systems since the basic assumptions are no longer valid. Specifically, elementary theory neglects the square of the first derivative in the beam curvature formula and provides no correction for the shortening of the moment-arm cause by transverse deflection. These two effects must be considered to analyze the large deflection.

Through the correction of deflected geometry and internal axial force, the proposed new approach is developed from the linearized beam theory. The solutions from the proposed approach are compared with exact solutions.

1. Introduction

The elastic field matrices for the small-deflection analysis are derived by assuming a linear system. However, when large-deflection are involved, the system becomes nonlinear, and the assumptions from elementary beam theory are no longer valid. Specifically, elementary theory neglects the square of the first derivative in the beam curvature formula and provides no correction for the shortening of the moment-arm caused by transverse deflections. Thus, for large loads, elementary theory for a simple cantilever beam can give deflections greater than the length of the beam.

Several investigators have discussed techniques for analyzing mechanical systems when deflections are large. Bisshopp and Drucker [1] obtained the relationships between the end loads and the displacements in the longitudinal and transverse directions for an inextensible cantilevered beam with the end loads. In their analytical study, the exact expression for the beam curvature of the

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elastic line is related to the arc-length and the slope of the deflected beam. Then, they assumed that the curvature of the beam is proportional to the bending moment, and the curvature at the loaded end is zero. The solution required that elliptic integrals be evaluated numerically. The solution for more complex beams are addressed by Frisch-Fay [3] again through the use of elliptic integrals.

The previous methods have used procedures based on either elliptic integrals or finite elements. Both methods require extensive computer resources (time or storage) when applied to complex mechanisms. The method presented here is based on transfer matrices, and is efficient in terms of both time and storage.

2. Correction for Large Deflection Analysis

2.1 Fundamentals for Analysis

Large-deflection problems cannot be solved directly using elementary beam theory, because the theory neglects both the square of the first derivative in the denominator of the beam curvature formula and the shortening of the moment-arm.

However, if these effects are evaluated approximately and involved iteratively, the large-deflection problems may be analyzed using linearized equations. The displacements for the linearized system are corrected by a geometric relationship, and an updated internal axial force in each segment is determined from equilibrium conditions. The corrections for the displacements and average axial force are updated at every iteration.

A general beam subjected to external loadings is represented in Fig. (1). Regardless of the beam loading, the beam can be accurately modeled as a series of discrete segments so that each segment is subjected to the internal forces at both ends as shown in Fig. (2). Each segment has its own local coordinate system oriented at an angle with respect to the fixed global system. The position of the local coordinate system must be updated as the member deforms.

A typical beam segment can be represented as shown in Fig. (3). The internal forces at both ends are present in the local coordinate system. As the segment deflects, the moment-arm is shortened by the transverse displacement due to both the transverse and axial loadings.

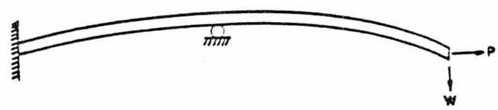


Fig. (1) A general beam subjected to external loadings

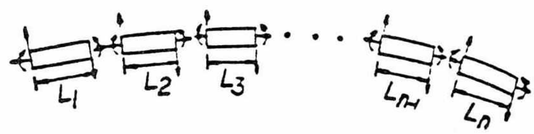


Fig. (2) Beam divided into finite segments

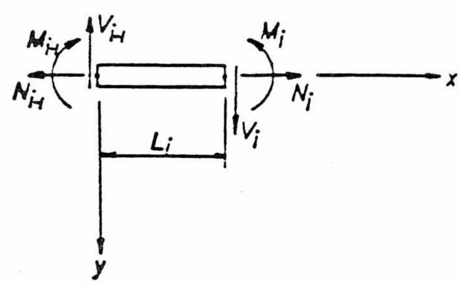


Fig. (3) Internal force in a segment

2.2 State Equations for the Linearized System

The relationships among the displacements and internal forces at both ends of a segment under axial tension for the linearized system can be derived based on classical beam theory and are given as follows :

$$u'_i = u'_{i-1} + \frac{L}{E \cdot A} N_{i-1} \quad \cdots\cdots (1)$$

$$w'_i = w'_{i-1} - \frac{\sinh KL}{K} \cdot \theta_{i-1} + \frac{1 - \cosh KL}{P} \cdot M_{i-1} + \left(\frac{L}{P} - \frac{\sinh KL}{KP}\right) \cdot V_{i-1} \quad \cdots\cdots (2)$$

$$\theta_i = \cosh KL \cdot \theta_{i-1} + \frac{K \cdot \sinh KL}{P} \cdot M_{i-1} + \frac{\cosh KL - 1}{P} \cdot V_{i-1} \quad \cdots\cdots (3)$$

$$M_i = \frac{P \cdot \sinh KL}{K} \cdot \theta_{i-1} + \cosh KL \cdot M_{i-1} + \frac{\sinh KL}{K} \cdot V_{i-1} \quad \cdots\cdots (4)$$

$$V_i = V_{i-1} \quad \cdots\cdots (5)$$

$$N_i = N_{i-1} \quad \cdots\cdots (6)$$

Where u' = longitudinal displacement in the linearized system,
w' = transverse displacement in the linearized system,
θ = slope,
M = moment,
V = internal force in the transverse direction,
N = internal force in the longitudinal direction,
P = axial force on the segment,
L = length of the segment,
and $K = \sqrt{P/EI}$.

2.3 Displacement Correction from Geometrical Constraints

The displacements u' and w' in Eqs. (1) and (2) are defined in the local coordinate system for the small deflection analysis as shown in Fig. (4). But for a large deflection analysis, the displacements are related to each other and to the total length of the segment($L+\triangle L$), which L is the original length of the segment and $\triangle L$ is the elongation due to the axial (tensile) force. Generally $\triangle L$ will be small and can be neglected. The angle α (see Fig.4) is the rotation angle between the current and original local coordinate systems. This angle must be updated iteratively as the segment deforms.

Fig. (4) shows the displacements at the end of the segment relative to the inclined axes, where the inclined axes are dependent on the deflected position of the beam end. Then, the displacements in the local coordinate system can be determined from the geometric relationship as follows :

$$u = u' \cdot \cos\alpha + w' \cdot \sin\alpha \quad \cdots\cdots (7)$$

$$w = -u' \cdot \sin\alpha + w' \cdot \cos\alpha \quad \cdots\cdots (8)$$

and $\alpha = \theta/2 \quad \cdots\cdots (9)$

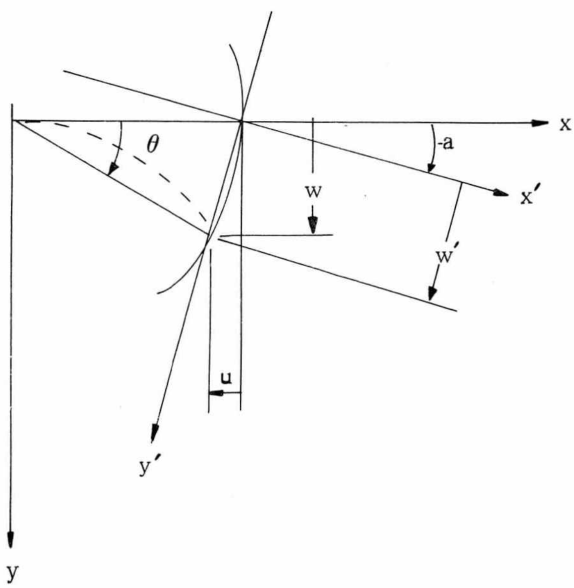


Fig. (4) Transverse displacement in an inclined axes

Where θ is the average slope of the deflected segment. In the actual situation, α is not exactly θ/2, because the deflected segment is not a straight line as shown in Fig. (4). But, if the length of the segment is short enough to be approximately straight, the relationship can be used for the formulations.

Substituting Eqs. (1) and (2) into Eqs, (7) and (8) gives

$$u_i = u_{i-1} - \frac{\sinh KL}{K}\sin\alpha\cdot\theta_{i-1} + \frac{1-\cosh KL}{P}\sin\alpha\cdot M_{i-1} + \left(\frac{L}{P}-\frac{\sinh KL}{K\cdot P}\right)\cdot\sin\alpha\cdot V_{i-1} + \frac{L}{EA}\cos\alpha\cdot N_{i-1} \quad (10)$$

$$w_i = w_{i-1} - \frac{\sinh KL}{K}\cos\alpha\cdot\theta_{i-1} + \frac{1-\cosh KL}{P}\cos\alpha\cdot M_{i-1} + \left(\frac{L}{P}-\frac{\sinh KL}{K\cdot P}\right)\cdot\cos\alpha\cdot V_{i-1} + \frac{L}{EA}\sin\alpha\cdot N_{i-1} \quad (11)$$

The other components of the state vector (θ, M, V, and N) are not changed.

2.4 Internal Axial Forces from Equilibrium Conditions

In the equations given above, K involves the axial force P. In the iterative analysis, it is convenient to assume that P is a constant during each individual step and then to correct the value for P between steps. Let us consider the equilibrium conditions of the segment as shown in Fig. (5) to determine a way of iteratively updating the average axial force (P) in each beam segment. The average axial force in the segment is determine so that equilibrium conditions at the beam ends can be satisfied.

Fig. (5) shows a beam segment under large deflections. The relationships for the state variables between nodes i and i−1 are given in Eqs. (3)−(6) and Eqs. (10)−(11). Here, the eqilibrium condition for the moments in the beam segment is investigated. From the summation of moment about the right end of the segment, the following condition is given :

$$\Sigma M = 0 = M_{i-1} - M_i + (L+\triangle u)\cdot V_{i-1} - (\triangle w)\cdot N_{i-1} \quad (12)$$

Here, Δu and Δw are the net displacements in the axial and transverse directions. The net displacements in each direction can be calculated from Eqs. (10) and (11) as follows :

$$\triangle u = u_i - u_{i-1} \quad (13)$$

$$\triangle w = w_i - w_{i-1} \quad (14)$$

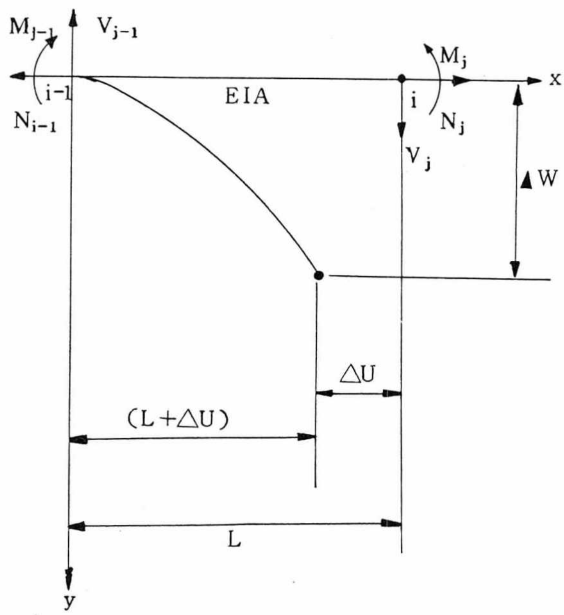


Fig.(5) A beam segment under large deflection

Combining Eqs. (4), (12), (13), and (14), and rearranging the results gives

$$[\frac{-\sinh KL}{K}\cdot\theta_{i-1}+\frac{1-\cosh KL}{P}\cdot M_{i-1}+(\frac{L}{P}-\frac{\sinh KL}{KP})\cdot V_{i-1}]\cdot(P+V_{i-1}\cdot\sin\alpha-N_{i-1}\cdot\cos\alpha)+\frac{L}{EA}\cdot N_{i-1}\cdot(V_{i-1}\cdot\cos\alpha+N_{i-1}\cdot\sin\alpha)=0 \quad \cdots\cdots (15)$$

The first square bracket in Eq. (15) corresponds to w' in Eq. (2) and this term cannot be zero. Here, the elongation or shortening of the segment length due to the axial force is very small relative to the displacement due to the transverse deflection. Then, the second terms in the above equation can be neglected. Thus, the internal axial force in a segment can be derived from the terms in Eq. (15) as follows :

$$P=N_{i-1}\cdot\cos\alpha-V_{i-1}\cdot\sin\alpha \quad \cdots\cdots (16)$$

2.5 Procedure for Iterative Transfer Matrix Method

Each link is divided into many sections with a lumped elastic stiffness and a lumped mass. The necessary transfer matrices at each section can be determined from the material properties, geometry, and external loads. After the transfer matrices for all of the sections and nodes are determined, a system equation is built by multiplying the matrices from the starting node to the end node. As in common boundary-value problems, the boundary conditions at each end point are applied so that there are three unknowns and three knowns in the state vectors at the starting point and the end point (usually supports). Three unknowns at the starting support can be easily determined by solving three linear simultaneous equations. Finally, all of the state vectors at every nodes in the system are calculated by multiplying the corresponding transfer matrix and the state vector of the previous node. At the initial iteration, the zero-axial force condition is used to determine the internal forces at any node of the flexible-body systems. Because the axial force at each section is unknown at the initial iteration, equilibrium cannot be represented explicitly so that the equations become non-linear. The moment at each segment depends on the transverse forces and displacements as well as on the unknown axial force. In addition, the displacements at the end of each segment is a function of the unknown internal forces. This means that a field matrix must incorporate the unknown axial loads which are part of the state variables. In this method, the nonlinear problem is linearlized by first separating the interrelative elastic effects for the segment into the transverse and the longitudinal directions, and by calculating the internal forces in each direction. For subsequent iterations, the field matrices are calculated again by using the internal forces determined during the previous iteration. The accuracy of the matrix improves with each iteration. Convergence of this method is very fast, and usually less than 10 iterations are required.

3. Comparison with Exact Solution

A cantilever beam under transverse loading at

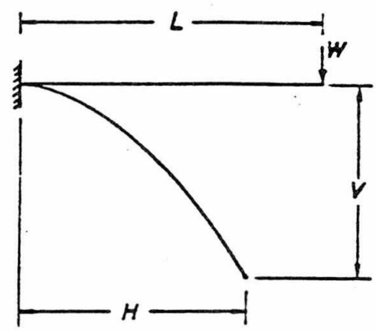
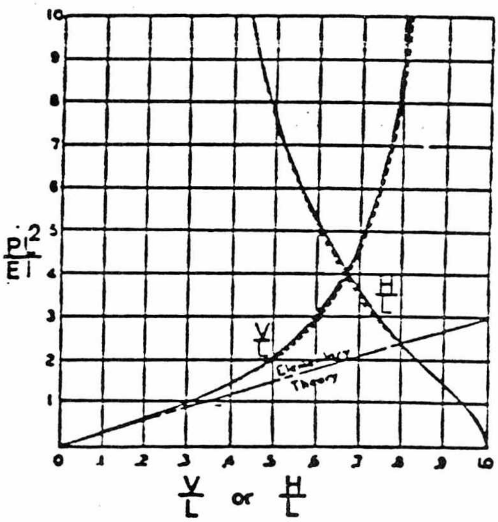


Fig.(6)A cantilever beam under transverse loading at tip



······Solution from the proposed method [4]
----Exact solution from bisshopp and drucker [1,2]
(b) Solutions of large-deflection problem

Fig.(7) Solutions of large deflection problem

tip [1] will be investigated. A steel cantilever beam shown in Fig. (6) has a 245 mm length with a square cross-section of 2.54 x 2.54 (mm^2). The total number of elements are 10 with each element of equal length. The results are compared with the curves presented in References [1,2] and given in Fig. (7). The comparative study shows that both curves for H/L and V/L are in good agreement.

4. Summary

The iterative transfer-matrix method is developed for the large deflection analysis of the flexible systems. The field matrices are derived by correcting the displacements for the linearized system from geometrical constraints and the internal axial force from equilibrium conditions. The comparative study with the exact solution shows that the results are in good agreement. The iteration transfer-matrix method converges rapidly, and fewer than 10 iterations are usually required.

『References』

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