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A Change Point Problem in the Regression Model When the Errors are Correlated+

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ABSTRACT

Testing procedures for a detection of change point in the regression model with correlated errors are discussed. A Bayesian approach is adopted and applied to a regression model with errors following an AR(1) model.

1. Introduction

Increasing interest has been shown in the change point problem. This problem of testing for the change at unknown time point was considered by Quandt(1958, 1960) who proposed a test procedure for no change versus one change based on the likelihood ratio. Brown, Durbin and Evans(1975) provided tests based on recursively generated residuals and Macneil(1978) proposed tests for the change of polynomial regression which are based on raw regression residuals. Based on the Bayesian approach, Chernoff and Zacks(1964), Hobert and Broemeling (1977), Smith and Cook(1980), and Booth and Smith(1982) studied similar problems by assigning a prior distribution to the change point, $k(1 \leqq k \leqq n)$.

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Most authors, however, have assumed that the observations are mutually independent, which is often unreasonable for sequential data.

For dependent series, Box and Tiao(1965, 1975) studied tests for changes where the possible change point is known, and Bagshaw and Johnson(1977) proposed a method for detecting step changes in AR(1) model. Henderson(1986) studied the change point problem in mean when the data are correlated and the variance of each element and correlation matrix are known. In this paper we will consider the change point problem in regression coefficients when the errors are correlated.

Consider the regression model

$$\underline{y}=\underline{x}\beta+\underline{j}_k\delta+\underline{e}, \tag{1.1}$$

where $\underline{y}$ is an $(n\times 1)$ column vector, $\underline{x}=(x_1, x_2, \cdots, x_n)'$ is a vector of regressor variables, $\underline{j}_k$ is given by $\underline{j}_k=(0, 0, \cdots, 0, x_{k+1}, x_{k+2}, \cdots, x_n)'$, β is an initial slope, δ is the amount of chage in β, and $\underline{e}$ is a normal random vector with mean $\underline{0}$ and correlation matrix R, all elements having variance σ^2. In other words, the sequences defined by (1.1) will be said to have a change point at k if

$$y_t=x_t\beta+e_t, \qquad \text{for } 1\leq t\leq k$$
$$y_t=x_t(\beta+\delta)+e_t, \qquad \text{for } k+1\leq t\leq n.$$

As in Henderson(1986), the correlation structure R and σ^2 are assumed to be known, the initial slope may or may be known, δ denoting the change of β is unknown but δ is assumed to be positive, and k is unknown. Note that since we know the correlation structure R transformation to independence is possible. But the design matrix after transformation will be complicated and does not have the step change structure, $[\underline{x}, \underline{j}_k]$, in (1.1) as Henderson(1986) stated. Therefore, when the observations are correlated we cannot use procedures studied for independent cases.

Testing and estimation procedures are described in Section 2 and 3. In Section 4, we discuss the case when R and δ^2 are unknown. Finally, in Section 5, we will test our procedure using correlated data and examin the effect of correlation.

2. Testing Procedure for a Change Point: R and σ^2 are Known

We begin our discussion of a Bayesian approach, assuming that the prior distribution for k is

$$P_0(n)=p, \quad P_0(k)=(1-p)/(n-1),$$

where $k=1, 2, \cdots, n-1$, $0\leq p\leq 1$. No change in β implies that the change point k is n.

For given k, β and δ, the likelihood function is given by

$$L(\underline{y}|k,\ \beta,\ \delta) = (2\pi\sigma^2)^{-n/2}|R|^{-\frac{1}{2}}\exp\left[-\frac{1}{2\sigma^2}(\underline{y}-\underline{x}\beta-\underline{j}_k\delta)'R^{-1}(\underline{y}-\underline{x}\beta-\underline{j}_k\delta)\right].$$

The likelihood ratio for testing $H_0 : k=n$ versus $H_1 : k<n$ is

$$\frac{(n-1)^{-1}\sum_{k=1}^{n-1} L(\underline{y}|k,\ \beta,\ \delta)}{L(\underline{y}|n,\ \beta)} \tag{2.1}$$

2.1 Case with β Known

In this section we assume that β is known, and, without loss of generality, let $\beta=0$. Given a change in slope β, we adopt the uniform prior distribution of k, $i.\ e.$,

$$P_0(k) = 1/(n-1),\quad k=1,\ 2,\ \cdots,\ n-1.$$

Denote the elements of R^{-1} by (r_{ij}) and introduce the notation

$$A_k = \underline{j}_k'R^{-1}\underline{j}_k,\quad B_k = \underline{j}_k'R^{-1}\underline{y},\quad r_{i.} = \sum_{j=1}^{n} r_{ij},\quad r_{..} = \sum_{i=1}^{n}\sum_{j=1}^{n} r_{ij}.$$

With $\beta=0$, (2.1) can be written by

$$\frac{(n-1)^{-1}\sum_{k=1}^{n-1} L(\underline{y}|k,\ \delta)}{L(\underline{y}|n)}$$

$$= \frac{1}{n-1}\sum_{k=1}^{n-1}\exp\left[-\frac{1}{2\sigma^2}(\delta^2\underline{j}_k'R^{-1}\underline{J}_k - 2\delta\underline{j}_k'R^{-1}\underline{y}\right]$$

$$= \frac{1}{n-1}\sum_{k=1}^{n-1}\exp\left[-\frac{1}{2\sigma^2}(\delta^2 A_k - 2\delta B_k)\right] \tag{2.2}$$

On Taylor expansion in δ, this ratio becomes

$$1 + \frac{1}{(n-1)\sigma^2}\left(\sum_{k=1}^{n-1} B_k\right)\delta + o(\delta).$$

We thus suggest the following test :

$$\text{reject } H_0 \text{ if } T_1 = \sum_{k=1}^{n-1} B_k > C.$$

Note that the test statistic T_1 can be written as

$$T_1 = \underline{u}'\underline{y} = \sum_{j=1}^{n} u_j y_j, \tag{2.3}$$

where $\underline{u}' = (u_1, u_2, \cdots, u_n)$ and u_j is given by

$$u_j = \sum_{i=1}^{n} (i-1) x_i r_{ij}.$$

Note also that if each element of $\underline{x}$ is equal to 1, T_1 becomes $\sum_{j=1}^{n}\sum_{i=1}^{n}(i-1)r_{ij}y_j$ which is the test statistic derived by Henderson(1986). If each element of $\underline{x}$ is 1 and the observations are independent, then the test statistic T_1 reduces to $\sum_{i=1}^{n}(i-1)y_i$ which was given by Chernoff and Zacks(1964).

Since T_1 is a linear combination of y_i's, T_1 is a normal random variable with variance $\sigma^2 \underline{u}'R\underline{u} = v^2$. Under H_0, $E(T_1)=0$, and the critical value for a size α test is given by $vc_{1-\alpha}$, where $c_{1-\alpha}$ is the $100(1-\alpha)$ percentile of the standard normal distribution. Under H_1,

$$\theta_k = E(T_1) = \delta \underline{u}'\underline{j}_k$$

and the power of the test is given by

$$1-\Phi(c_{1-\alpha} - \theta_k / v) \tag{2.4}$$

2.2 Case with β Unknown

Now, consider the case when the initial slope β is unknown. In this case, we consider the following methods for the manipulation of β for given k and δ.

(i) For each fixed k, find the maximum likelihood estimation(M. L. E) of β and use it in the likelihood function.

(ii) Assuming a $N(0, \tau^2)$ prior for β, integrate out the likelihood function with respect to β, under H_0 and H_1, and then let τ^2 go to infinite.

(iii) Giving an improper prior $P_0(\beta) \propto |I(\beta)|^{\frac{1}{2}}$ to β, where I is the Fisher's information number, integrate out the likelihood function with respect to β, under H_0 and H_1.

For each case, we obtain the following results.

(i)' The M. L. E. of β under H_1 at fixed k is given by

$$\hat{\beta}_k = (\underline{x}'R^{-1}\underline{x})^{-1}\underline{x}'R^{-1}(\underline{y} - \underline{j}_k\delta)$$

and under H_0

$$\hat{\beta} = (\underline{x}'R^{-1}\underline{x})^{-1}\underline{x}'R^{-1}\underline{y}.$$

The likelihood ratio(2.1) with the above M. L. E.'s of β becomes

$$\frac{1}{n-1}\sum_{k=1}^{n-1} \exp\Big[-\frac{1}{2\sigma^2}\{[\underline{j}_k'R^{-1}(I-\underline{x}(\underline{x}'R^{-1}\underline{x})^{-1}\underline{x}'R^{-1})\underline{j}_k]\delta^2$$

$$-2\delta\underline{j}_k'R^{-1}(\underline{y}-\underline{x}\hat{\beta})\}\Big],$$

which is similar to (2.2). Hence on Taylor expansion in δ, we get the test statistic T_2,

$$T_2=\sum_{k=1}^{n-1}\underline{j}_k'R^{-1}(\underline{y}-\underline{x}\hat{\beta})$$

$$=\underline{u}'(\underline{y}-\underline{x}\hat{\beta}),$$

which is a weighted sum of residuals obtained assuming no change in β with weights u_i's.

(ii)′ Assuming that the prior of β is $N(0, \tau^2)$, the likelihood ratio (2.1) is given by

$$\frac{\frac{1}{n-1}\sum_{k=1}^{n-1}\int_{-\infty}^{\infty}L(\underline{y}|k,\ \beta,\ \delta)P_0(\beta)\,d\beta}{\int_{-\infty}^{\infty}L(\underline{y}|n,\ \beta)P_0(\beta)\,d\beta}$$

$$=\frac{1}{n-1}\sum_{k=1}^{n-1}\exp\Big[-\frac{1}{2\sigma^2}\{[\underline{j}_k'R^{-1}\underline{j}_k-(\tau^4(\underline{j}_k'R^{-1}\underline{x})^2)/(\tau^2(\tau^2(\underline{x}'R^{-1}\underline{x})+\sigma^2))]\delta^2$$

$$-2[\underline{j}_k'R^{-1}\underline{y}-\tau^4(\underline{j}_k'R^{-1}\underline{x})(\underline{x}'R^{-1}\underline{y})/(\tau^2(\tau^2(\underline{x}'R^{-1}\underline{x})+\sigma^2))]\}\delta\Big].$$

Again, the Taylor expansion in δ gives the test statistic

$$\sum_{k=1}^{n-1}\Big[\underline{j}_k'R^{-1}\underline{y}-\frac{\tau^4(\underline{j}_kR^{-1}\underline{x})(\underline{x}'R^{-1}\underline{y})}{\tau^2(\tau^2(\underline{x}'R^{-1}\underline{x})+\sigma^2)}\Big].$$

Now, if τ^2 goes to infinite, the above statistic becomes T_2 as in (i)′.

(iii)′ Adopting $P_0(\beta)\propto[(\sigma^{-2})\underline{x}'R^{-1}\underline{x}]^{\frac{1}{2}}$ as the prior of β and integrating out the likelihood function with respect to β under H_0 and H_1, the likelihood ratio becomes

$$\frac{1}{n-1}\sum_{k=1}^{n-1}\exp\Big[-\frac{1}{2\sigma^2}\{\delta^2[(\underline{j}_k'R^{-1}\underline{j}_k)-(\underline{j}_k'R^{-1}\underline{x})^2(\underline{x}'R^{-1}\underline{x})^{-1}]$$

$$-2\delta[\underline{j}_k'R^{-1}\underline{y}-\underline{j}_k'R^{-1}\underline{x}(\underline{x}'R^{-1}\underline{x})^{-1}\underline{x}'R^{-1}\underline{y})]\}\Big].$$

As before, we can see that the coefficient of δ after Taylor expansion in δ is equal to T_2.

Whatever priors are used, following the procedure just described, we can obtain the same test statistic T_2. T_2 is also a normal random variable such that, under H_0, $E(T_2)=0$ and

$$\gamma^2 = \mathrm{var}(T_2) = \sigma^2 \underline{u}'(R - \underline{x}(\underline{x}'R^{-1}\underline{x})^{-1}\underline{x}')\underline{u},$$

and under H_1

$$\mu_k = E(T_2) = \delta \underline{u}'(I - \underline{x}(\underline{x}'R^{-1}\underline{x})^{-1}\underline{x}'R^{-1})\underline{j}_k.$$

Note that if each element of $\underline{x}$ is equal 1, T_2 reduces to $\underline{u}'(\underline{y} - \bar{y}^*\underline{1}_n)$, where $\bar{y}^* = (\sum_{i=1}^{n} r_{i.} y_i)/r_{..}$, as derived by Henderson(1986) and, moreover, when the observations are independent, T_2 becomes $\sum_{i=1}^{n}(i-1)(y_i - \bar{y})$ which was acquired by Chernoff and Zacks(1964).

The power of T_2 is given by

$$1 - \Phi(c_{1-\alpha} - \mu_k/\gamma). \tag{2.5}$$

2.3 Prior Specification for β and δ

Booth and Smith(1982) considered the similar problem when the errors are independent. Using the vague prior for β, δ, and σ, they derived a statistic to test whether change occurs at time k. We assume the improper prior for β and δ as follows :

Under H_1, for fixed k, we assume that

$$P_0(\beta, \delta | k) \propto |I(\beta, \delta)|^{\frac{1}{2}}$$
$$= \sigma^{-2}\left[(\underline{x}'R^{-1}\underline{x})(\underline{j}_k'R^{-1}\underline{j}_k) - (\underline{j}_k'R^{-1}\underline{x})^2\right]^{\frac{1}{2}}, \tag{2.6}$$

where

$$I(\beta, \delta) = \sigma^{-2}\begin{bmatrix} \underline{x}'R^{-1}\underline{x} & \underline{j}_k'R^{-1}\underline{x} \\ \underline{j}_k'R^{-1}\underline{x} & \underline{j}_k'R^{-1}\underline{j}_k \end{bmatrix}.$$

and under H_0,

$$P_0(\beta) \propto |I(\beta)|^{\frac{1}{2}} = \sigma^{-1}(\underline{x}'R^{-1}\underline{x})^{\frac{1}{2}}.$$

Then the appropriate ratio is given by

$$T_3 = \frac{\frac{1}{n-1}\sum_{k=1}^{n-1}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} L(\underline{y}|k, \beta, \delta) P_0(\beta, \delta|k)\, d\beta d\delta}{\int_{-\infty}^{\infty} L(\underline{y}|n, \beta) P_0(\beta)\, d\beta}$$

$$= C \frac{1}{n-1} \sum_{k=1}^{n-1} \exp\left\{ \frac{[(\underline{x}'R^{-1}\underline{x})(\underline{j}_k'R^{-1}\underline{y}) - (\underline{j}_k'R^{-1}\underline{x})(\underline{x}'R^{-1}\underline{y})]^2}{2\sigma^2(\underline{x}'R^{-1}\underline{x})[(\underline{j}_k'R^{-1}\underline{j}_k)(\underline{x}'R^{-1}\underline{x}) - (\underline{j}_k'R^{-1}\underline{x})^2]} \right\},$$

where C is a constant which can be determined following the argument provided is Spiegelhalter and Smith (1982, Section 4).

If we assume the independence of β and δ, and use

$$P(\beta|k) = C_1[\sigma^{-2}(\underline{x}'R^{-1}\underline{x})]^{\frac{1}{2}}$$

and

$$P(\delta|k) = C_2[\sigma^{-2}(\underline{j}_k'R^{-1}\underline{j}_k)]^{\frac{1}{2}},$$

then we obtain

$$T_3' = \frac{C_2}{n-1} \sum_{k=1}^{n-1} \left[\frac{2\pi(\underline{j}_k'R^{-1}\underline{j}_k)(\underline{x}'R^{-1}\underline{x})}{(\underline{j}_k'R^{-1}\underline{j}_k)(\underline{x}'R^{-1}\underline{x}) - (\underline{j}_k'R^{-1}\underline{x})^2} \right]^{\frac{1}{2}}$$

$$\times \exp\left\{ \frac{[(\underline{x}'R^{-1}\underline{x})(\underline{j}_k'R^{-1}\underline{y}) - (\underline{j}_k'R^{-1}\underline{x})(\underline{x}'R^{-1}\underline{y})]^2}{2\sigma^2(\underline{x}'R^{-1}\underline{x})[(\underline{j}_k'R^{-1}\underline{j}_k)(\underline{x}'R^{-1}\underline{x}) - (\underline{j}_k'R^{-1}\underline{x})^2]} \right].$$

For each k, we can derive a test statistic and a testing procedure using normal or χ^2 distribution. But, an overall assesment of change versus no change is difficult because we have not been able to derive the distribution of T_3 or T_3'. The same is true with Booth and Smith (1982). For each fixed k they derived a test statistic of the form given by a weighted average of the Bayes factor in (4. 3), of which the distribution is hard to obtain. However, the choice of prior distribution does not give much effect on the form of the statistics.

2.4 Multiple Parameter Case

We now turn to the more general case,

$$\underline{y} = X\underline{\beta} + J_k\underline{\delta} + \underline{e},$$

where $X = (\underline{x}_1, \underline{x}_2, \cdots, \underline{x}_p)$, $J_k = (\underline{j}_1, \underline{j}_2, \cdots, \underline{j}_p)$, $\underline{\beta} = (\beta_1, \beta_2, \cdots, \beta_p)'$, $\underline{\delta} = (\delta_1, \delta_2, \cdots, \delta_p)'$, and $\underline{y}$ and $\underline{e}$ are given by (1. 1). If we use a $N(\underline{o}, I\tau^2)$ prior for $\underline{\beta}$ in method (ii) or $P_0(\underline{\beta}|k) \propto [\sigma^{-2}(X'R^{-1}X)]^{\frac{1}{2}}$ in method (iii), we obtain the similar results to the method (i) as in the single parameter case. Therefore, we consider the method (i) only.

For given k and δ, the M. L. E of $\underline{\beta}$ is, under H_1,

$$\hat{\underline{\beta}}_k = (X'R^{-1}X)^{-1}X'R^{-1}(\underline{y} - J_k\underline{\delta})$$

and under H_0

$$\underline{\hat{\beta}} = (X'R^{-1}X)^{-1}X'R^{-1}\underline{y}.$$

The likelihood ratio is therefore

$$\frac{(n-1)^{-1}\sum_{k=1}^{n-1} L(\underline{y}|k,\ \underline{\hat{\beta}}_k,\ \underline{\delta})}{L(\underline{y}|n,\ \underline{\hat{\beta}})}$$

$$= \frac{1}{n-1}\sum_{k=1}^{n-1} \exp\Big[-\frac{1}{2\sigma^2}\{\underline{\delta}'J_k'R^{-1}(I - X(X'R^{-1}X)^{-1}X'R^{-1})J_k\underline{\delta}$$

$$-2\underline{\delta}'J_k'R^{-1}(I - X(X'R^{-1}X)^{-1}X'R^{-1})\underline{y}\}\Big].$$

On Taylor expansion in $\underline{\delta}$, the ratio bacomes

$$1 + \frac{1}{(n-1)\sigma^2}\sum_{k=1}^{n-1} J_k'R^{-1}(I - X(X'R^{-1}X)^{-1}X'R^{-1})\underline{y} + o(\underline{\delta}).$$

Hence the test statistic, T_4 say, is given by

$$T_4 = \sum_{k=1}^{n-1} J_k'R^{-1}(I - X(X'R^{-1}X)^{-1}X'R^{-1})\underline{y}.$$

T_4 is a normal random variable which is similar to T_2 except using matrix notation for $\underline{x}$ and $\underline{j}_k$.

3. Estimation of a Change Point

Consider the estimation problem of change point k by Bayesian approach. A point which maximizes the posterior density function will be regarded as the change point. Smith (1975, 1977) studied this problem using Bayesian method when the errors are independent and σ^2 is unknown.

Following Smith, we can obtain the posterior distribution

$$P(k|\underline{y},\ \beta,\ \delta) \propto P(\underline{y}|k, \beta, \delta) P_0(\beta,\ \delta|k)\ P_0(k), \tag{3.1}$$

where we adopt the uniform prior for k, *i.e.*, $P_0(k) = \frac{1}{n-1}$, $k=1,\ 2,\ \cdots,\ (n-1)$, and the improper prior $P_0(\beta,\ \delta|k)$ of β and δ is given by (2.6). Since

$$P(k|\underline{y}) \propto \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} P(\underline{y}|k,\ \beta,\ \delta)\ P_0(\beta,\ \delta|k)\ P_0(k)\ d\beta d\delta$$

$$\propto \exp \frac{1}{2\sigma^2(\underline{x}'R^{-1}\underline{x})} P_k,$$

where

$$P_k = \frac{[(\underline{x}'R^{-1}x)(\underline{j}_k'R^{-1}\underline{y}) - (j_k'R^{-1}\underline{x})(\underline{x}'R^{-1}\underline{y})]^2}{(\underline{x}'R^{-1}x)\ (\underline{j}_kR^{-1}\underline{j}_k) - (\underline{j}_k'R^{-1}\underline{x})^2},$$

the posterior mode of k is the point which maximizes P_k. For fixed k, if we denote the residual sum of squares of the model $\underline{y}=\underline{x}\beta+\underline{j}_k\delta+\underline{e}$ and $\underline{y}=\underline{x}\beta+\underline{e}$ by SSE(k) and SSE(n), respectively, then

$$\text{SSE}(k) = \underline{y}'R^{-1}(I - Z_k'(Z_k'R^{-1}Z_k)^{-1}Z_k'R^{-1})\underline{y} \tag{3.2}$$

and

$$\text{SSE}(n) = \underline{y}'R^{-1}(I - \underline{x}'R^{-1}\underline{x})^{-1}\underline{x}'R^{-1})\underline{y}, \tag{3.3}$$

where $Z_k = (\underline{x} : j_k)$. After some calculation we can see that P_k is equal to SSE(n) − SSE(k). That is to say, the point that minimizes SSE(k) is our estimate of change k since SSE(n) is constant with respect to k. As in Section 2, without difficulty we can obtain the similar results for multiple regression model.

4. Testing Procedure for a Change Point: R and σ^2 are Unknown

We have assumed that R and σ^2 are known, which may be true in some situation when the long series of observations are available. But in many other cases, it is impractical to assume that R and σ^2 are known. We first consider the case when σ^2 is unknown but R is known.

As in Section 2.2, for the manipulation of β and σ, we adopt the improper prior for β and σ

$$P(\beta,\ \sigma) \propto |I(\beta,\ \sigma)|^{\frac{1}{2}}$$

$$= \sigma^{-2}\ (\underline{x}'R^{-1}\underline{x})^{\frac{1}{2}}$$

Then, given k and δ, the ratio in (2.1) becomes

$$\frac{\frac{1}{n-1}\sum_{k=1}^{n-1}\int_0^\infty\int_{-\infty}^\infty L(\underline{y}|k,\ \beta,\ \delta,\ \sigma)P_0(\beta,\ \sigma)\,d\beta d\sigma}{\int_0^\infty\int_{-\infty}^\infty L(\underline{y}|n,\ \beta,\ \sigma)P_0(\beta,\ \sigma)\,d\beta d\sigma}$$

$$=\frac{1}{n-1}\sum_{k=1}^{n}[1+2\delta C_k+\delta^2 D_k]^{-\frac{n}{2}},$$

where

$$C_k=\frac{\underline{j}_k'R^{-1}\underline{x}(\underline{x}'R^{-1}\underline{x})^{-1}\underline{x}'R^{-1}\underline{y}}{(\underline{y}'R^{-1}\underline{y})-\underline{y}'R^{-1}\underline{x}(\underline{x}'R^{-1}\underline{x})^{-1}\underline{x}'R^{-1}\underline{y}}$$

and

$$D_k=\frac{\underline{j}_k'R^{-1}\underline{j}_k-\underline{j}_k'R^{-1}\underline{x}(\underline{x}'R^{-1}\underline{x})^{-1}\underline{x}'R^{-1}\underline{j}_k}{\underline{y}'R^{-1}\underline{y}-\underline{y}'R^{-1}\underline{x}(\underline{x}'R^{-1}\underline{x})^{-1}\underline{x}'R^{-1}\underline{y}}$$

On Taylor expansion in δ of (4.1), the ratio becomes

$$1-\frac{n}{n-1}\sum_{k=1}^{n-1}C_k\delta+o(\delta)$$

and the appropriate test statistic is defined by

$$T_5=\sum_{k=1}^{n-1}C_k. \tag{4.2}$$

Since C_k is a normal random variable divided by a χ^2 distributed random variable, the exact distribution of T_5 is hard to obtain as in T_3'.

As was explained in Section 2.3, when $R=I$, Booth and Smith(1982) adopted the vague prior for β, δ and σ such that, under H_1,

$$P_0(\beta,\ \delta,\ \sigma)=P(\beta|\sigma)P(\delta|\sigma)P(\sigma)$$

$$=C_\beta(2\pi\sigma^2)^{-\frac{p}{2}}\ C_\delta(2\pi\sigma^2)^{-\frac{p}{2}}\ C_\sigma\sigma^{-1}$$

and under H_0

$$P_0(\beta,\ \delta)=C_\beta(2\pi\sigma^2)^{-\frac{p}{2}}\ C_\sigma\sigma^{-1}$$

where C_β, C_δ, C_σ are unknown constants. They proposed a test statistic for fixed k defined by

$$B_k=C_\delta\left[\frac{|X_n'X_n|}{|X_k'X_k|\ |X_{n-k}'X_{n-k}|}\right]^{\frac{1}{2}}[1+(p/(n-2p))F_k]^{n/2}\ , \tag{4.3}$$

where $X_k' = (x_1, x_2, \cdots, x_k)$, $X_{n-k}' = (x_{k+1}, \cdots, x_n)$, p is the number of regressor variables, and F_k denotes an usual F statistic for testing $\delta = 0$ at fixed point k. If we follow their approach but with improper priors for β, δ, and σ such that under H_1, for fixed k,

$$P(\beta, \delta, \sigma | k) \propto |I(\beta, \delta, \sigma)|^{\frac{1}{2}}$$

$$= \sigma^{-3} \left[(\underline{x}'R^{-1}\underline{x})(\underline{j}_k'R^{-1}\underline{j}_k) - (\underline{j}_k'R^{-1}\underline{x})^2 \right]^{\frac{1}{2}}$$

and under H_0 ,

$$P_0(\beta, \sigma) \propto \sigma^{-2} (\underline{x}'R^{-1}\underline{x})^{\frac{1}{2}},$$

the ratio in(2.1) reduces to

$$= C^* \left[\frac{SSE(n) - (SSE(n) - SSE(k))}{SSE(n)} \right]^{-n/2}$$

$$= C^* \left[1 + \frac{\mathrm{SSE}(n) - \mathrm{SSE}(k)}{\mathrm{SSE}(k))} \right]^{n/2}$$

$$= C^* (1 + F_k)^{n/2}, \tag{4.4}$$

where SSE(k) and SSE(n) are given by (3.2) and (3.3), respectively, and C^* can be determined following Booth and Smith(1982). We thus obtain the usual F statistic for testing $\delta = 0$ at fixed point k.

In case when R and σ^2 are unknown, the estimation of k may be achieved by giving priors to all parameters including R. Especially, if R is a function of only one parameter ϕ as in the AR(1) case, (5.1), $P_0(\phi)$ can be used as in Henderson(1986). In general, however, we face with two problems as Henderson stated : Firstly, how to find suitable estimators or priors for R and σ^2 and secondly, how to derive the distributions of test statistics.

These problems need further work.

5. Simulation Study

In this section, we examine our test procedure and study the effect of serial correlation using simulated data.

The test statistic T_1 acquired in simple regression model with no intercept is used, assuming σ^2 and R are known. In our simulation, the error e_t's satisfying the AR(1) model

$$e_t = \phi e_{t-1} + a_t \tag{5.1}$$

are considered, where a_t's are independent and normally distributed with mean 0 and variance $\sigma_a^2=9$, and a_t' are generated from GGNML in IMSL and multiplied by 3. Without loss of generality $\beta=0$ is assumed, and thus y_t's are generated from our regression model

$$\underline{y}=\underline{j}_k\delta+\underline{e}.$$

We compare the powers of T_1, changing the values of AR(1) parameter(ϕ) from -0.4 to 0.4 with the number of observations 20, $x_i=i/4\,(1\leq i\leq n)$, and a change point $k=15$.

Exact powers at significance level 0.05 with various values of ϕ are given in Figure 1 when σ^2 is $\sigma_a^2/(1-\phi^2)$ and δ is $\sigma/6$.
Recall that the power is given by

$$1-\Phi(c_{1-\alpha}-\theta_k/v)$$

which is given in (2.4). Note that the elements of the inverse of AR(1) correlation matrix are

$$r_{ij}=\begin{bmatrix} 1/(1-\phi^2) & \text{if} & i=j=1,\ n \\ -\phi/(1-\phi^2) & \text{if} & |i-j|=1 \\ (1+\phi^2)/(1-\phi^2) & \text{if 1} & 1<i=j<n \\ 0 & \text{o.w.} & \end{bmatrix}$$

Thus, the weight u_j's are given as follows :

$$u_1=-\phi x_2/(1-\phi^2)$$

$$u_j=\sum_{i=1}^{n}(i-1)x_i r_{ij}$$

$$=(1-\phi^2)^{-1}\{(j-1)[(1+\phi^2)x_j-\phi(x_{j-1}+x_{j+1})]+\phi(x_{j-1}-x_{j+1})\} \qquad (5.2)$$

for $j=2,\ 3,\ \cdots,\ n-1$ with $x_0=x_{n+1}=0$

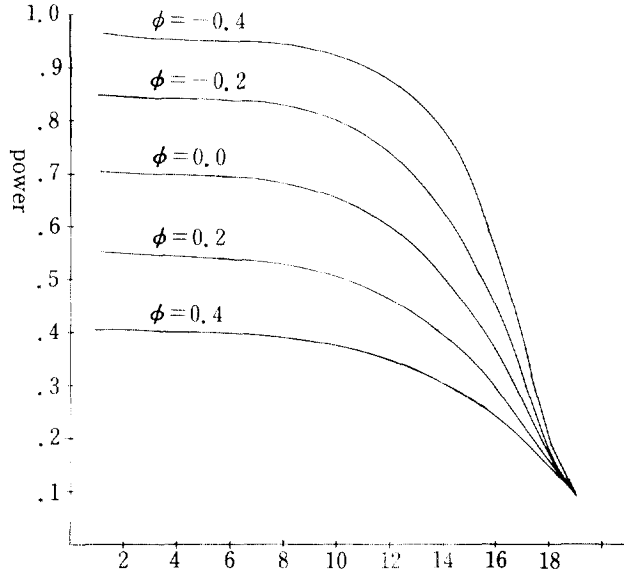
$$u_n=[-(n-2)\phi x_{n-1}+(n-1)x_n]/(1-\phi^2).$$

When all elements of $\underline{x}$ are 1, θ_k/v becomes

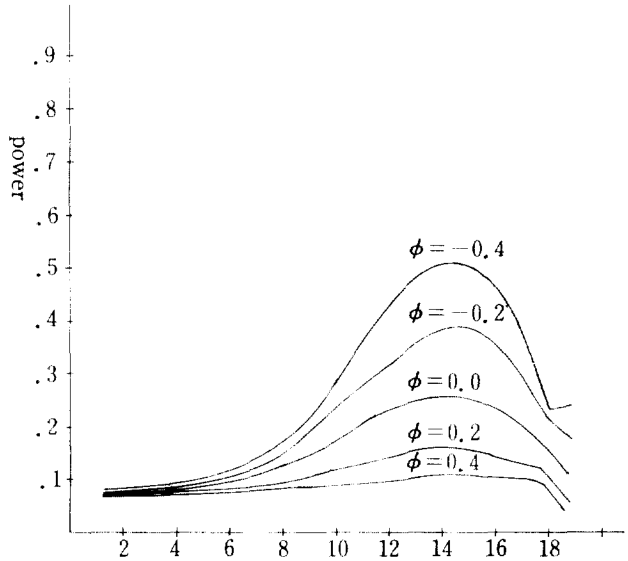
$$\frac{\delta\sum_{j=i+1}^{n}(j-1)}{\sigma_a\sqrt{\sum_{i=1}^{n}\sum_{j=1}^{n}(i-1)\phi^{|j-i|}(j-1)}}$$

Hence the test is more powerful for negatively large ϕ and for small k as we can see in Figure 1.

The power comparision using T_2 is given in Figure 2 under the same condition except using $\hat{\beta}$ for β as in T_1. A similar explanation can be given for this case but the explicit form for μ_k/γ in (2.5) is too complicated and is omitted.



Power of T_1 Test for $n=20$ and $\delta=\sigma/6$,
When Testing at the 5% Level and Assuming $\beta=0$.
(Figure 1)



Power of the T_2 Test for $n=20$ and $\delta=\sigma/6$,
When Testing at the 5% Level and Assuming β Unknown
(Figure 2)

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