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INTEGRAL REPRESENTATION OF GENERALIZED INVARIANT REFLECTION POSITIVE MATRIX KERNEL

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Abstract: Using the method of eigenfunction expantion of self adjoint operators, we establish the necessary and sufficient conditions for a reflection positive generalized matrix kernel to be represented in an integral form. This integral converges in the cosidered spaces.

1. Introduction

It is shown that every positive definite invariant matrix on a Hilbert space has a certain integral form (see [7, 8]). In this paper, we show that the reflection positivity is a necessary and sufficient condition for the invariant matrix $K=(K_{jk})_{j,\,k=0}^{\infty}$, $K_{jk}\in(s'(R^n))^{j+k}$ has an integral form.

Reflection positivity was introduced in quantum field theory by K. Osterwalder and R. Schrader [13]. They showed that it is the necessary and sufficient conditions for the distribution $G_n\varepsilon S'(R^{4n})$ (Green's functions) to be a Fourier Laplace transform of a positive measure of not too fast increase (see [12, 13]).

The method used here is the eigenfunction expansion of a system of self adjoint operators [1]. Ju.M. Berezanskii and others used this method for obtaining the integral representation of positive definite functions and kernels [1, 2]. I.M. Gali and others used the same method for representing the positive definite invariant matrix kernel $K=(K_{jk})_{j,\,k=0}^{\infty}$ and Wightman functional in integral form [7, 8, 9].

Consider the rigged Hilbert spaces

$$H_-\supseteq H_0\supseteq H_+ \tag{1}$$

with the involution $\omega\to\bar{\omega}$ defined in H_- and also in H_0 and H_+. Construct a sequence of chains of a separable Hilbert spaces with the involution

$$\phi'^j\supseteq\cdots\supseteq\mathscr{H}^j_{-n}\supseteq\cdots\supseteq\mathscr{H}^j_0\supseteq\cdots\supseteq\mathscr{H}^j_n\supseteq\cdots\supseteq\phi^j \tag{2}$$

where $H^j=H\otimes\cdots\otimes H$ stands for the tensor product j copies of the Hilbert space $H(J=0, 1, 2\cdots$, for $j=0$, $H^0=C'$, where C' is thecomplex field, and

$$(\phi')^j = \bigcup_{n=0}^{\infty} \mathscr{H}_n^j = (\phi^j)'$$

is the space of generalized vectors. $\mathscr{H}_{-n} = \mathscr{H}'_n$ (see [1, 11]).

Let Π be the set of all finite sequences

$$\hat{u} = (u_0, u_1, \cdots)(u_j \varepsilon \phi^j).$$

A generalized matrix kernel K is the matrix $\hat{K} = (K_{jk})_{j,k=0}^{\infty}$ consisting of all generalized vectors $K_{jk} \varepsilon (\phi')^{j+k}$.

A generalized matrix kernel $\hat{K} = (K_{jk})_{j,k=0}^{\infty}$ is said to be reflection positive (R. P.) if for any finite sequence $(u_j)_{j=0}^{\infty} (u_j \varepsilon \phi^j)$

$$\sum_{j,k=0}^{\infty} (K_{jk}, u_j \otimes \theta u_k^*) H_0^{j+k} \geq 0 \tag{3}$$

where. * is the involution defined by

$$u_k^*(x_1, x_2, \cdots, x_k) = u(x_k, x_{k-1}, \cdots, x_1)$$

and the reflection θ given by

$$\theta u_k(x_1, x_2, \cdots, x_k) = u_k(-x_1, \cdots, -x_k).$$

Let G be a locally compact commutative Lie group and $g \to U_g$ be a representation of a group G in ϕ. The matrix kernel $\hat{K} = (K_{jk})_{j,k=0}^{\infty}$ is said to be invariant relative to this representation if

$$(\underbrace{U'_g \otimes \cdots \otimes U'_g}_{j+k}) K_{jk} = K_{jk} \tag{4}$$

where $g \to U'_g$ is the adjont representation. If this representation is irreducible then the kernel $\hat{K}$ is said to be elementary [2, 5].

LEMMA. *In order that the R. P. generalized matrix kernel $\hat{K} = (K_{jk})_{j,k=0}^{\infty}$, $K_{jk} \varepsilon (\phi')^{j+k}$, be invariant (in the sense of 14), it is necessary and sufficient to have the following representation*

$$K_{jk} = \int_{R^n} \Omega_{jk}(\theta; \lambda) d_\rho(\lambda), \quad (j, k = 0, 1, \cdots) \tag{5}$$

$\lambda = (\lambda_1, \cdots, \lambda_n) \varepsilon R^n$, where $\hat{\Omega}(\theta; \lambda) = (\Omega_{jk}(\theta; \lambda))_{j,k=0}^{\infty}$, $\Omega_{jk}(\theta; \lambda) \varepsilon (\phi')^{j+k}$, be a system of elementary R. P. matrix kernels, $d_\rho(\lambda)$ a non-negative finite measune, and the integral converges weakly. Conversely, every integral in the form (5) is R. P. and translation invariant.

OUTLINE OF PROOF. Define the convergence $\hat{u}^n \to \hat{u}$ in the topology of Π by $u_m^n = 0$ for $m > N$ and $u_k^n \to u_k$ in ϕ^k, $(k=0, 1, \cdots, N)$.

Analogous to (1), we construct

$$\Pi' \supseteq \hat{H}_0 \supseteq \Pi \qquad . \tag{6}$$

Then, the R.P. kernel $\hat{K} \varepsilon \Pi' \otimes \Pi'$, and is invariant relative to the representation $g \to U_g$ of the group G in Π' by the formula

$$\hat{U}_g \hat{u} = (u_0, U_g u_1, (U_g \otimes U_g) u_2, \cdots)$$

The system of R.P. matrix kernels $\hat{\Omega}(\theta; \lambda)$ form a family of elementary R. P. matrix kernel i.e., for every λ, we have

$$\|\Omega_{jk}(\lambda; \theta)\|_{\substack{j+k \\ -m_{jk}}} \leq C_{ik} < \infty$$

and

$$-i(A_v^* \otimes \underbrace{I \otimes \cdots \otimes I}_{j+k-1} + \cdots + \underbrace{I \otimes I \otimes \cdots \otimes I}_{j-1} \otimes A_v^* \otimes \underbrace{I \otimes \cdots \otimes I}_{k}) \times \times \Omega_{jk}(\theta; \lambda)$$
$$= \lambda \Omega_{jk}(\lambda; \theta)$$
$$i(\underbrace{I \otimes \cdots \otimes I}_{j} \otimes A_v^* \otimes \underbrace{I \otimes \cdots \otimes I}_{k-1} + \underbrace{I + \cdots + I \otimes \cdots \otimes I}_{j+k-1} \otimes A_v^*) \Omega_{jk}(\lambda; \theta) = \lambda \Omega_{ik}(\lambda; \theta),$$
$$(v = 1, 2, \cdots, n)$$

According to theorems of Borchers and Araki(see [3] for references to the original papers, see also [6]). If the operators A_v^* on Π are restricted to self-adjoint operators in $\mathscr{H}_{\hat{k}}$ and in order to construct the integral (5), we shall give the decomposition of the spaces $\mathscr{H}_{\hat{k}}$ into a continuous direct sum of Hilbert spaces $\mathscr{H}_{\Omega(\lambda; \theta)}$

$$\hat{\mathfrak{L}}_k = \int_{R^n} \mathscr{H}_{\Omega(\lambda; \theta)} d_\rho(\lambda)$$

where $d_\rho(\lambda)$ is a probability non-negative finite measure defined uniquely by $\hat{\mathfrak{L}}_k$.

Let G be a group of translation in R^n, $\phi = S(R^n)$ be the space of test functions which are infinitely differentiable on R^n and together with all derivatives are rapidly decreasing as $|x| \to \infty$.

Let the generalized matrix $\hat{K} = (K_{jk})_{j, k=0}^{\infty}$, $K_{jk} \varepsilon (S^{j+k}(R^n))$, be R.P. and invariant relative to the group of translations in R^n; then the linear transformation

$$(x_1, \cdots, x_j) \to (x, \xi_1, \cdots, \xi_{j-1})$$

where $x=J^{-1}(x_1+\cdots+x_j)$, $\xi_1=(x_1-x_2)$, $\cdots$, $\xi_{j-1}=(x_{j-1}-x_j)$ maps $\hat{K}$ to some R.P. matrix and

$$(\bar{u}_0,\ \overline{u_1(x_1)},\ \cdots,\ \overline{u_j(x_j,\ \cdots,\ x_1)},\ \cdots)\rightarrow(\bar{u}_0,\ u_1(x),\ \cdots,\ u_j(x_j-\xi_j,\ \cdots,\ \xi_1),\ \cdots)$$

$$\hat{u}=(u_0,\ \cdots,\ u_j(x,\ \xi_1,\ \cdots,\ \xi_{j-1}),\ \cdots)\varepsilon\Pi$$

In order to formulate the basic result of this paper we need the concept of non-negative matrix measure $dR(\lambda)$ with values in $(\phi^{j+k})'$, $(j,\ k=1,\ 2,\ \cdots)$.

This is a matrix function of Borel sets Δ of R^n, $R(\Delta)=(R_{jk}(\Delta))_{j,\ k=0}^{\infty}(R_{00}(\Delta)\varepsilon C'$, $R_{0k}\varepsilon(\phi')^{k-1}$, $R_{j0}\varepsilon(\phi')^{j-1})$. It is R.P. matrix measure in the sense of (3) if with the chain (2) we can find a sequence of Hilbert spaces

$$\mathscr{H}_{n_1}^{j},\ \mathscr{H}_{n_2}^{j},\ \cdots,\ \mathscr{H}_{n_j}^{j}$$

such that

$$R_{0k}(\Delta)\varepsilon\mathscr{H}_{-n_k}^{k-1},\quad R_{j0}(\Delta)\varepsilon\mathscr{H}_{-n}^{j-1},\quad R_{jk}(\Delta)\varepsilon\mathscr{H}_{-n_n}^{j-1}\otimes\mathscr{H}_{-n_k}^{k-1}$$

$\hat{R}(\Delta)$ is R.P. for all $\Delta\varepsilon B(R^n)$ and countably additive (for $j,\ k=0,\ 1,\ \cdots$, we have $R_{jk}(\bigcup_i^{\infty}\Delta_i)=\sum_{i=1}^{\infty}R_{jk}(\Delta_i)$ if the sets $\Delta_i\varepsilon R^n$ are disjoint. The series being convergent in the norm of $(\phi^{j+k})'$.

Since $\phi=S(R^n)$ is nuclear space then in the chain (2) we can find a sequence of Hilbert spaces $\mathscr{H}_{n_1}^{j}$, $\mathscr{H}_{n_2}^{j}$, $\cdots$, $\mathscr{H}_{n_j}^{j}$ [10] such that the corresponding kernel $R_{jj}(\Delta)$ in the spaces $\mathscr{H}_{-n_j}^{2j-2}$ for the operators $C_j(\Delta)$ have a finite trace

$$\mu_i(\Delta,\ u_j)=tr_{\mathscr{H}_{-n_j}^{j-1}}C_j(\Delta)=tr_{\mathscr{H}_{-n_j}^{2j-2}}R_{ij}(\Delta),\quad \Delta\varepsilon R^n$$

We say that the matrix $\hat{R}(\Delta)$ is a tempered measure if the numerical measure $\mu_j(\Delta,\ m_j)$ satisfies the condition

$$\int_{R^n}(1+|\lambda|^2)^{j-p_j}d\mu_j(\lambda,\ n_j)\leq\infty \text{ for some } p_j\geq 0$$

From all the above we have the following result:

THEOREM. *A R.P. generalized matrix* $\hat{K}=(K_{jk})_{j,\ k=0}^{\infty}$, $K_{jk}\varepsilon(S'(R^n))^{j+k}$, *is invariant relative to the group of translations in* R^n *if and only if it admits a representation*

$$K_{jk}=\int_{R^n}e^{i\lambda(,\ y-\theta x)}dR_{jk}(\xi_1,\ \cdots,\ \xi_{j-1},\ \eta_1,\ \cdots,\ \eta_{k-1},\ \lambda)$$

where $\hat{R}(\Delta)=(R_{jk}(\Delta))_{j,\ k=0}^{\infty}$ *is R.P. matrix tempered measure, the elements* R_{jk}

for $j=0$ or $k=0$ are concentrated at the point $\lambda=0$ and the measure $\hat{R}(\Delta)$ is defined uniquely by the matrix $\hat{K}$.

COROLLARY(1). *By using the R.P. matrix kernel $\hat{K}=(K_{jk})_{j,k=0}^{\infty}$, we construct a matrix $L=(L_{jk})_{j,k=0}^{\infty}$, $L_{jk}\varepsilon S'(R^n))^{j+k}$ by the relation*

$$(L_{jk}, v_j \otimes u_k^{*}) = (K_{jk}, v_j \otimes \theta \dot{u}_k^{*})$$

$$i.e., \; L_{jk}(x_1, x_2, \cdots, x_j, y_1, \cdots, y_k) = K_{jk}(-x_j, \cdots, -x_1, y_1, \cdots, y_k).$$

From translation invariant and reflection positivity of $\hat{K}=(K_{jk})_{j,k=0}^{\infty}$ the matrix $L=(L_{jk})_{j,k=0}^{\infty}$ is translation invariant and positive definite and then we can find the result of [9].

COROLLARY(2). *By analytic continuation which replaces λ by-$i\lambda$ we have the Feynman-Kac formula for general potential of the kernel $K=(K_{jk})_{j=1}^{\infty}$, $K_{jk}\varepsilon(S^{j+k}(R^n))'$ (see* [12]).

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