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A Study on Periodic Semigroups

By Younki Chae

Let S denote a topological semigroup throughout: that is, S is a Hausdorff space with a continuous associative multiplication, denoted by juxtaposition.

S is said to be pointwise periodic if and only if for each $x\in S$, $x^p=x$ for some integer $p\geq 2$. The least such p will be called the period of x.

S is turned periodic if and only if there is an integer $n\geq 2$ such that $x^n=x$ for all $x\in S$. The least such n will be called the period of S.

S is said to be recurrent if and only if x is a limit point of $\{x^n|n\geq 2\}$ for all $x\in S$.

Clearly S is pointwise periodic if and only if S is the union of finite groups. When S is compact, recurrency is equivalent to S being the union of groups (Cliffordian) [5, 6].

The criteria for S being pointwise periodic and recurrent could be found in [5, 6] as follows:

(1) S is pointwise periodic if and only if for every subset A of S,

$$A^2\subset A \text{ implies } A^2=A.$$

(2) A compact S is recurrent if and only if for each compact subset K of S,

$$K^2\subset K \text{ implies } K^2=K.$$

Many questions on pointwise periodic semigroups and recurrent semigroups were raised by A.D. Wallace in [10]. Here are striking results by J. M. Day [3, 4]:

(A) If S is locally compact totally disconnected pointwise periodic, and $x^n=x$, then x has an arbitrary small compact open neighborhood V such that $V^n=V$.

(B) If S is compact connected recurrent and x is a cut point of S, then $x^n=x$ for some $n\geq 2$.

(C) If S is locally compact totally disconnected recurrent, and $x^n=x$, then x has an arbitrary small compact open neighborhood W such that $W^n=W$.

As indicated in [1] and [2], A pointwise periodic semigroup seems closely

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related to a semilattice. The aim of this paper is to find conditions that a pointwise periodic semigroup to be a semilattice. Also, various properties of pointwise periodic semigroups were inrestigated.

An element e of S is called an idempotent if and only if $e^2=e$. The set of all idempotents of S is denoted by $E(S)$. The set of idempotents of a semigroup may be empty, as in the case for the additive semigroup of positive integers. However, $E(S)$ is non-empty if S is compact. Moreover, in any topological semigroup S, $E(S)$ is closed. For e, $f\in E(S)$, define $e\leq f$ if and only if ef=e =fe. Then $\leq$ is a partial order on E and is a closed subspace of $S\times S$.

THEOREM 1. *Let S be pointwise periodic and let p be the period of $x\in S$. Then*

(1) $E(S)=\{x^{p-1}|x\in S\}$

(2) *If $x^\alpha=x$, then $\alpha=m(p-1)+1$ for some positive integer m.*

(3) *If $p-1$ is a prime number, then the period of $x^\alpha (\alpha<p-1)$ is the same as the period p of x.*

(4) *Let p and q be the periods of the elements x and y of S respectively and let l be the least common multiple of $p-1$ and $q-1$. If S is commutative, then $(xy)^{l+1}=xy$.*

PROOF. (1) If $p=2$, then $x^2=x=x^{p-1}\in E(S)$. If $p>2$, then

$$(x^{p-1})^2=x^{2p-2}=x^p x^{p-2}=xx^{p-2}=x^{p-1},$$

and hence $x^{p-1}\in E(S)$.

Now let $y\in E(S)$, *i.e.*, $y^2=y$. Then $y\in\{z, z^2, \cdots, z^{r-1}\}$ for some $z\in S$ with period r. Let $y=z^\alpha$, $1\leq\alpha\leq r-1$. Since $y^2=y$, $z^\alpha=z^{2\alpha}$. Then

$$z=z^r=z^{r-\alpha}z^\alpha=z^{r-\alpha}z^{2\alpha}=z^{r+\alpha}=z^{\alpha+1}.$$

But $\alpha+1\leq r$, and therefore $\alpha+1=$r, i.e., $\alpha=r-1$.

(2) Since $x^\alpha=x$, $p\leq\alpha$. Let $\alpha=m(p-1)+r$, where m is a positive integer and $0\leq r<p-1$. By (1), one obtain

$$x=x^\alpha=x^{m(p-1)}x^r=x^{p-1}x^r=x^{p-1+r}.$$

Then $p-1+r<2p-1$ gives $r=1$. Hence $\alpha=m(p-1)+1$.

(3) The period of x^α is less than or equal to p since $(x^\alpha)^p=(x^p)^\alpha=x$. Suppose $(x^\alpha)^n=x^\alpha$ for some integer n $(1<n<p)$. Then

$$x=x^\alpha x^{p-\alpha}=x^{n\alpha}x^{p-\alpha}=x^{(n-1)\alpha+1}.$$

By(2), $(n-1)\alpha=m(p-1)$ for some positive integer m. Since $p-1$ is a prime number, $p-1|n-1$ or $p-1|\alpha$ which are both impossible. Therefore p is the period of x^{α}.

(4) Let $l=(p-1)m=(q-1)n$. Then

$$(xy)^{l+1}=x^{(p-1)m}y^{(q-1)n}xy=x^{p-1}y^{q-1}xy=xy.$$

REMARK. In (3), x^{α} may not have period p for all $(1\leq\alpha<p-1)$ if $p-1$ is not a prime. For example, let $p=5$. Then $(x^2)^3=xx^5=xx=x^2$, i.e., x^2 has period 3. In (4), $l+1$ may not be the period of xy. For example, let $Z_6=\{0, 1, \cdots, 5\}$ be the semigroup under the multiplication modulo 6. Then Z_6 is a discrete pointwise periodic semigroup. In Z_6, the periods of 0, 1, 2, 3, 4, 5 are 2, 2, 3, 2, 2, 3 respectively. The least common multiple of (period of 2)-1 and (period of 5)-1 is 2, i.e., $l+1=3$. However, $2.5\equiv 4 \pmod 6$ and the period of 4 is 2.

THEOREM 2. *Let S be commutative and pointwise periodic. Define a relation $\leq$ on S by $x\leq y$ if and only if $x^2=xy$. Then*

(1) $\leq$ is a closed partial order on S.

(2) $L(x)\equiv\{y\,|\,y\leq x\}=E(S)x$.

(3) If $e\in E(S)$ and $x\in S$, then $xe\leq x$.

(4) If $a\leq b$ and $x\leq y$, then $ax\leq by$.

PROOF. (1) Let $x\leq y$ and let p be the period of x. Then $x^2=xy$. If $p>2$, $x=x^{p-2}x^2=x^{p-2}xy=x^{p-1}y$. If $p=2$, $x^{p-1}y=xy=x^2=x$. Conversely, $x=x^{p-1}y$ implies $x^2=xx^{p-1}y=xy$, i.e., $x\leq y$. Therefore $x\leq y$ if and only if $x=x^{p-1}y$.

1) Since $x^2=xx$, $x\leq x$, i.e., $\leq$ is reflexive.

2) Let $x\leq y$ and $y\leq x$ and let p and q be periods of x and y respectively. Then $x=x^{p-1}y$ and $y=y^{q-1}x$, and hence

$$y=y^{q-1}x=y^{q-1}x^{p-1}y=x^{p-1}y=x.$$

That is, $\leq$ is anti-symmetric.

3) Let $x\leq y$ and $y\leq z$. Then $x^2=xy$, $y^2=yz$, and $x=x^{p-1}y$. Hence

$$xz=x^{p-1}yz=x^{p-1}y^2=(x^{p-1}y)y=xy=x^2,$$

i.e., $x\leq z$. Therefore $\leq$ is transitive, and hence $\leq$ is a partial order on S.

Now let f, g: $S\times S\rightarrow S$ by $f(x, y)=x^2$ and $g(x, y)=xy$. Then f and g are continuous, and

$$\leq\ =\{(x, y)\,|\,x^2=xy\}=\{(x, y)\,|\,f(x, y)=g(x, y)\}$$

is closed since S is a Hausdorff space.

(2) Let $y \in E(S)x$. Then $y=ex$, $e^2=e$. Hence

$$y^2=(ex)^2=x(ex)=xy, \text{ i.e., } y \leq x.$$

Therefore $y \in L(x)$.

Now let $y \in L(x)$. Then $y \leq x$, i.e., $y=y^{q-1}x$, where q is the period of y. By (1) in Theorem 1, $y=y^{q-1}x \in E(S)x$, and we are done.

(3) Since $(xe)^2=(xe)x$, $xe \leq x$, $^{\forall} e \in E(S)$.

(4) Let $a \leq b$ and let $c \in S$. Then $a^2=ab$, and

$$(ac)^2=a^2c^2=abc^2=(ac)(bc),$$

i.e., $ac \leq bc$. Now let $x \leq y$. Then $ax \leq bx \leq by$.

REMARK. If S has a zero 0, then $0 \leq x$ for all $x \in S$ since $0^2=0=0x$. If S has an identity 1, $e \leq 1$ for every $e \in E(S)$ since $e^2=e=e\,1$.

THEOREM 3. *Let $\leq$ be the partial order on the commutative pointwise periodic S defined in Theorem 2. Let m be a positive integer. If $xy \leq x^m$, y^m, $^{\forall}x$, $y \in S$, then S is a semilattice.*

PROOF. By hypothesis, $(xy)^2=(xy)y^m$, $^{\forall}x$, $y \in S$. The substitution of x^{p-1}, where p is the period of x, for y gives

$$x^2=(xx^{p-1})^2=x(x^{p-1})^{m+1}=xx^{p-1}=x.$$

Hence S is a semilattice.

THEOREM 4. *Let S be commutative and pointwise periodic. Then S admits a partial order $\leq$ such that $xy \leq x$, y, $^{\forall}x$, $y \in S$*
if and only if S is a semilattice.

PROOF. Let $x \in S$. By substituting x, x^2, $\cdots$, for y in $xy \leq y$, one obtain

$$x \geq x^2 \geq x^3 \geq \cdots$$

Let p be the period of x. Then $x \geq x^2 \geq x^p = x$, i.e., $x^2=x$. Hence $S=E(S)$.

The converse is well known.

COROLLARY. *Let S be a commutative pointwise periodic semigroup which is not a semilattice. Then there is no partial order on S such that $xy \leq x$, y, $^{\forall}x$, $y \in S$.*

Let E be a quasi-ordered set. Then $X \subset E$ is said to be convex if and only if

$$a \leq b \leq c,\ a,\ c \in X \text{ implies } b \in X.$$

Let E be a topological space equipped with a quasi-order. The topology of E

is said to be locally convex if and only if the set of convex neighborhood of every point of E is a base for the neighborhood system of this point.

LEMMA. *Nachbin* [7] *Every compact partially ordered space is locally convex.*

THEOREM 5. *Let S be locally compact and recurrent. If S admits a partial order $\leq$ such that*

$$xy \leq x,\ y,\ {}^{V}x,\ y \in S.$$

Then for each open set U containing $x \in S$, $x^2 \in U$.

PROOF. By hypothesis, we have

$$x \geq x^2 \geq x^3 \geq \cdots.$$

Let V be an open subset of S with compact closure such that $x \varepsilon V \subset \overline{V} \subset U$. Since S is recurrent, $x^p \in V$ for some integer $p \geq 2$. Since $\overline{V}$ is a compact partially ordered space, by the above Lemma, $\overline{V}$ is locally convex. Then

$$x,\ x^p \in V,\ x \leq x^2 \leq x^p$$

implies $x^2 \in V \subset U$.

THEOREM 6. *Let S be pointwise periodic. Then S is periodic if and only if there is an integer $m \geq 2$ such that*

$$xy^m = x^m y,\ {}^{V}x,\ y \in S.$$

PROOF. Let p be the period of S. Then $p \geq 2$ and

$$xy^p = xy = x^p y,\ {}^{V}x,\ y \in S.$$

Now let x be any element of S and let p be the period of x. Then, by hypothesis,

$$x = xx^{p-1} = x(x^{p-1})^m = x^m x^{p-1} = x^m.$$

Hence S is periodic with period $\leq m$.

COROLLARY. *A commutative pointwise periodic semigroup S is a semilattice if and only if*

$$xy^2 = x^2 y,\ {}^{V}x,\ y \in S.$$

If x and Y are Hausdorff spaces and $\sigma : Y \to X$ is a continuous function, then $X \times S \times Y$ is a topological semigroup under the multiplication defined by

$$(x,\ s,\ y)\ (u,\ t,\ v) = (x,\ s\sigma(y,\ x)t,\ v).$$

This semigroup is called the Rees product of S over X and Y with sandwich function σ, and will be denoted by $[X, S, Y]_\sigma$. The Rees product of a topological group will be called a paragroup. In general, a paragroup fails to be a group. If σ is the constant function such that $\sigma(y, x)=1$, then

$$(x, s, y)^n=(x, s^n, y).$$

Hence we have the following two theorems immediately.

THEOREM 7. *If S is a pointwise periodic semigroup with* 1, *then $[X, S, Y]_\sigma$ is pointwise periodic.*

THEOREM 8. *If S is a recurrent semigroup with* 1, *then $[X, S, Y]_\sigma$ is recurrent.*

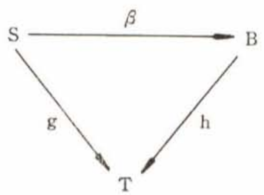
A semigroup S is said to be divisible if and only if for each $y\in S$ and $n\in N$, there exists $x\in S$ such that $x^n=y$.

It is clear that the surmorphic image of a divisible semigroup is divisible, and that the cartesian product of divisible semigroups is divisible. If S is a commutative finite divisible semigroup, then S is a semilattice.

THEOREM 9. *Let S be commutative and periodic. If S is divisible, then S is a semilattice.*

PROOF. Let p be the period of S. Define a function $f: S\to S$ by $f(x)=x^{p-1}$. Then $f(S)\subset E(S)$. Since S is divisible, f is surjective. Hence $S=E(S)$.

A Bohr compactification of a topological semigroup S is a pair (β, B) such that B is a compact semigroup, $\beta: S\to B$ is a continuous homomorphism, and if $g: S\to T$ is a continuous homomorphism of S into a compact semigroup T, then there exists a unique continuous homomorphism $h: B\to T$ such that the diagram:



commutes.

For any topological semigroup S, there exist a unique Bohr compactification

(β, B) of S up to isomorphism. Moreover, $\beta(S)$ is a dense subset of B.

THEOREM 10. *If S is periodic with period p, then the Bohr compactification of S is also periodic with period $\leq p$.*

PROOF. Let (β, B) be a Bohr compactification of S. Then $\beta(S)$ is a dense subsemigroup of B which is periodic with period $\leq p$. Let $b \in B$. Then there is a net $\{a_\alpha\}$ in $\beta(S)$ such that $a_\alpha \to b$. By the continuity of the multiplication in B, $a_\alpha^p \to b^p$. Then one obtain $b^p = b$.

As indicated in (A), if S is locally compact pointwise periodic and totally disconnected, each $x \in S$ has arbitrary small open neighborhood U such that $U^p = U$, where $x^p = x$.

THEOREM 11. *Let S be locally compact and totally disconnected and let S' be a subsemigroup of S. If S is pointwise periodic and if $x \in S'$, then x has arbitrary small open neighborhood A in S' such that $A^p = A$, where $x^p = x$.*

PROOF. Let $x \in S'$, $x^p = x$. If W is an open neighborhood of x in S', then there is an open neighborhood U of x in S such that $W = U \cap S'$. By the property (A) on S, there is an open set V in S containing x such that $V^p = V$. Now let $A = S' \cap V$. Then A is an open set in S containing x and

$$A = S' \cap V \subset S' \cap U = W,$$
$$A^p = (S' \cap V)^p \subset S'^p \cap V^p \subset S' \cap V \subset A.$$

Since S' is also pointwise periodic, $A^p = A$.

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