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PAIRS OF CONNECTIONS COMPATIBLE WITH ALMOST QUASI-QUATERNION STRUCTURES

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A.P. Norden [5] has introduced the notion of the mixed covariant differentiation by means of which a number of authors e.g. G. Atanasiu [1], A. Bonome and R. Castro-Bolano [2], A. Bucki and A. Miernowski [3], have defined pairs of connections compatible with certain structures on manifolds.

In this paper we define a pair of linear connections compatible with an almost quasi-quaternion structure on a manifold M. It turns out that every linear connection Γ on a manifold M determines a pair of linear connections compatible with an almost quasi-quaternion structure on M.

DEFINITION 1. [4]. Let M be a differentiable manifold of dimension $n=4m$ and assume that there is a 3-dimensional vector bundle Q consisting of tensor fields of type (1, 1) over M satisfying the following condition: In any coordinate neighbourhood U of M there is a local basis $\{F, G, H\}$ of Q such that:

(1) $F^2=G^2=-H^2=-Id,\ FG=GF=-H$

(2) $GH=HG=F,\ FH=HF=G$

where Id denotes the identity tensor field of type (1, 1) on M. Such a local basis is called a canonical basis of a bundle Q in U. Then, the bundle Q is called an almost quasi-quaternion structure on M and (M, Q) is an almost quasi-quaternion manifold.

DEFINITION 2. [4]. Suppose that M is an almost quasi-quaternion manifold with a structure $Q=\{F, G, H\}$. A linear connection Γ on M given by its covariant derivative ∇ is said to preserve a structure Q or simply to be a Q-connection if it satisfies:

$$\begin{aligned} \nabla F &= m_F\otimes Id+n_F\otimes F+p_F\otimes G+q_F\otimes H \\ \nabla G &= m_G\otimes Id+n_G\otimes F+p_G\otimes G+q_G\otimes H \\ \nabla H &= m_H\otimes Id+n_H\otimes F+p_H\otimes G+q_H\otimes H \end{aligned} \tag{3}$$

where m_J, n_J, p_J, q_J are certain 1-forms on M and J is F, G or H.

PROPOSITION 1. [4]. A linear connection Γ on an almost quasi-quaternion manifold M is a Q-connection if and only if:

(4) $$\nabla F=\nabla G=\nabla H=0$$

In [4] we have introduced the following tensor fields of type (2, 2):

(5) $$A=\frac{1}{4}(3Id\otimes Id+F\otimes F+G\otimes G-H\otimes H)$$

(6) $$B=\frac{1}{4}(Id\otimes Id-F\otimes F-G\otimes G+H\otimes H)$$

REMARK1. The operations of A (or$B$) on tensor fields: C of type (2,2), D of type (1, 2), F of type (1, 1) and X of type (1, 0) are expressed locally as follows: $A^{ij}_{kl}C^{ml}_{in}$, $A^{ij}_{kl}D^{1}_{mi}$, $A^{ij}_{k1}F^{1}_{l}$, $A^{ij}_{k1}X^{k}$ respectively.

The operators A and B have the following properties:

(7) $$A+B=Id\otimes Id,\quad A^2=A,\quad B^2=B,\quad AB=BA=0$$

(8) $$AF=AG=AH=0,\quad BId=Id,\quad BF=F,\quad BG=G,\quad BH=H$$

THEOREM 1. [4]. *The general family of Q-connections on a manifold M with an almost quasi-quaternion structure* $Q=\{F, G, H\}$ *is given by*:

(9) $$\overline{\nabla}_x=\nabla_x-\frac{1}{4}(F\nabla_x F+G\nabla_x G-H\nabla_x H)+BW_x$$

where ∇ *is a covariant derivative with respect to arbitrary initial linear connection* Γ *on* M *and* W *is any tensor field of type*(1,2) *with* $W_x Y=W(X, Y)$.

The curvature tensor field R_{XY} of a Q-connection has the following properties:

(10) $$FR_{XY}=R_{XY}F,\quad GR_{XY}=R_{XY}G,\quad HR_{XY}=R_{XY}H$$

(11) $$AR_{XY}=0$$

(12) $$BR_{XY}=R_{XY}$$

Suppose that on a manifold M, two linear connections $\overset{1}{\Gamma}$ and $\overset{2}{\Gamma}$ are given by means of their covariant derivatives $\overset{1}{\nabla}$ and $\overset{2}{\nabla}$ respectively. Following [5], we define the following mixed covariant derivatives for functions, vector fields and tensor fields of type (1, 1):

(13) $$\overset{12}{\nabla}_x f=XF=\overset{21}{\nabla}_x f$$

(14) $$\overset{12}{\nabla}_x Y=\overset{1}{\nabla}_x Y,\quad \overset{21}{\nabla}_x Y=\overset{2}{\nabla}_x Y$$

(15) $$(\overset{12}{\nabla}_x J)(Y, \omega)=X(J(Y, \omega))-J(\overset{1}{\nabla}_x Y, \omega)-J(Y, \overset{2}{\nabla}_x \omega)$$

(16) $$(\overset{21}{\nabla}_x J)(Y, \omega)=X(J(Y, \omega))-J(\overset{2}{\nabla}_x Y, \omega)-J(Y, \overset{1}{\nabla}_x \omega)$$

where f is a function, ω is a 1-form, X, Y are vector fields and J is a tensor field of type (1, 1) on M.

For a pair of connections $\overset{1}{\Gamma}$ and $\overset{2}{\Gamma}$ we define a mean connection $\overset{m}{\Gamma}$ given by its covariant derivative $\overset{m}{\nabla}$ and a deformation tensor field τ of type (1, 2) of these connections in the following form:

$$\overset{m}{\nabla}_x=\frac{1}{2}(\overset{1}{\nabla}_x+\overset{2}{\nabla}_x) \tag{17}$$

$$\tau_x=\overset{2}{\nabla}_x-\overset{1}{\nabla}_x \tag{18}$$

We may regard a tensor field J of type (1, 1) as a linear mapping of TM into TM defined as follows:

$$X\longrightarrow J(X,\omega)=\omega(J(X)),\ \omega \text{ is a 1-form on } M \tag{19}$$

In virtue of (19), (15) and (16) we have:

$$\omega((\overset{12}{\nabla}_x J)Y)=\omega((\overset{2}{\nabla}_x J+J\tau_x)Y),\ \ \omega((\overset{21}{\nabla}_x J)Y)=\omega((\overset{1}{\nabla}_x J-J\tau_x)Y)$$

for all 1-forms ω and vector fields X, Y. Hence we define:

$$\overset{12}{\nabla}_x J=\overset{2}{\nabla}_x J+J\tau_x \tag{20}$$

$$\overset{21}{\nabla}_x J=\overset{1}{\nabla}_x J-J\tau_x \tag{21}$$

We also have the following relations [3]:

$$\tau_x(J)=\tau_x J-I\tau_x \tag{22}$$

$$\overset{m}{\nabla}_x J=\frac{1}{2}(\overset{12}{\nabla}_x J+\overset{21}{\nabla}_x J) \tag{23}$$

$$\overset{1}{\nabla}_x(JK)=(\overset{21}{\nabla}_x J)K+J\overset{12}{\nabla}_x K \tag{24}$$

$$\overset{2}{\nabla}_x(JK)=(\overset{12}{\nabla}_x J)K+J\overset{21}{\nabla}_x K \tag{25}$$

for any tensor fields J, K of type (1, 1).

DEFINITION 3. A pair of linear connections $(\overset{1}{\Gamma}, \overset{2}{\Gamma})$ on M is said to be compatible with an almost quasi-quaternion structure $Q=\{F, G, H\}$ on M if:

$$\begin{aligned} \overset{12}{\nabla}_x F&=a_F(X)Id+b_F(X)F+c_F(X)G+d_F(X)H \\ \overset{12}{\nabla}_x G&=a_G(X)Id+b_G(X)F+c_G(X)G+d_G(X)H \\ \overset{12}{\nabla}_x H&=a_H(X)Id+b_H(X)F+c_H(X)G+d_H(X)H \end{aligned} \tag{26}$$

where a_J, b_J, c_J, d_J are certain 1-forms on M.

PROPOSITION 2. A pair of linear connections $(\overset{1}{\Gamma}, \overset{2}{\Gamma})$ on M is compatible with an almost quasi-quaternion structure $Q=\{F, G, H\}$ on M if and only if:

$$\begin{aligned}(27)\qquad &\overset{21}{\nabla}_x F=-\overset{12}{\nabla}_x F=b(X)Id-a(X)F-d(X)G+c(X)H\\ &\overset{21}{\nabla}_x G=-\overset{12}{\nabla}_x G=c(X)Id-d(X)F-a(X)G+b(X)H\\ &\overset{21}{\nabla}_x H=-\overset{12}{\nabla}_x H=-d(X)Id-c(X)F-b(X)G-a(X)H\end{aligned}$$

where a, b, c, d are certain 1-forms on M.

PROOF. From (24), if we put $J=K=F$ and make use of (1) we obtain: $(\overset{21}{\nabla}_x F)F+F(\overset{12}{\nabla}_x F)=0$ or $\overset{21}{\nabla}_x F=F(\overset{12}{\nabla}_x F)F$, hence because of (26) we have $\overset{21}{\nabla}_x F=F$ $[a_F(X)Id+b_F(X)F+c_F(X)G+d_F(X)H]F=-\overset{12}{\nabla}_X F$. Similarly, we show that: $\overset{21}{\nabla}_x G=-\overset{12}{\nabla}_x G$ and $\overset{21}{\nabla}_x H=-\overset{12}{\nabla}_x H$. Now, from (24) we have: $\overset{1}{\nabla}_x F=\overset{1}{\nabla}_x (GH)=(\overset{21}{\nabla}_x G)H$ $+G\overset{12}{\nabla}_x H=[-a_G(X)Id-b_G(X)F-c_G(X)G-d_G(X)H]H+G[a_H(X)Id+b_H(X)F+c_H$ $(X)G+d_H(X)H]=[-d_G(X)-c_H(X)]Id+[d_H(X)-c_G(X)]F+[-b_G(X)+a_H(X)]G$ $+[-a_G(X)-b_H(X)]H$.

Similarly, $\overset{1}{\nabla}_x G=[-d_F(X)-b_H(X)]Id+[c_F(X)+a_H(X)]F+[-b_F(X)+d_H(X)]G$ $+[-a_F(X)-c_H(X)]H$ and $\overset{1}{\nabla}_x H=[c_F(X)-b_G(X)]Id+[-d_F(X)+a_G(X)]F+[-a_F$ $(X)+d_G(X)]G+[b_F(X)-c_G(X)]H$. On account of Definition 2, a connection $\overset{1}{\Gamma}$ is a Q-connection so in virtue of Proposition 1 we have:

$$d_H=c_G=b_F=a,\ c_H=-d_G=-a_F=b,\ b_H=-a_G=-d_F=c,\ a_H=b_G=c_F=d$$

REMARK 2. From this Proposition it follows that the notion of compatibility of a pair $(\overset{1}{\Gamma}, \overset{2}{\Gamma})$ with a structure $Q=\{F, G, H\}$ is symmetric with respect to these connections.

THEOREM 2. *A pair of linear connections* $(\overset{1}{\Gamma}, \overset{2}{\Gamma})$ *on a manifold* M *is compatible with an almost quasi-quaternion structure* $Q=\{F, G, H\}$ *on* M *if and only if*:

$$(28)\qquad \overset{1}{\nabla}_x=\nabla_x-\frac{1}{4}(F\nabla_x F+G\nabla_x G-H\nabla_x H)+BP_x$$

$$\begin{aligned}(29)\qquad \overset{2}{\nabla}_x=\nabla_x-\frac{1}{4}(F\nabla_x F+G\nabla_x G-H\nabla_x H)+BP_x+a(X)Id\\ +b(X)F+c(X)G+d(X)H\end{aligned}$$

where ∇ *is a covariant differentiation operator with respect to arbitrary linear connection* Γ *on* M, P *is arbitrary tensor field of type* $(1,2)$ *with* $P_Y(X)=P(Y,$

X), *B is given by* (6) *and a, b, c and d are certain l-forms on M.*

PROOF. In the proof of Proposition 2, we have shown that $\overset{1}{\Gamma}$ is a Q-connection on M and from (20) $\overset{2}{\nabla}_x J = -\overset{1}{\nabla}_x J$, so $\overset{2}{\Gamma}$ is also a Q-connection on M. Hence

$$\overset{1}{\nabla}_x F = \overset{1}{\nabla}_x G = \overset{1}{\nabla}_x H = 0, \quad \overset{2}{\nabla}_x F = \overset{2}{\nabla}_x G = \overset{2}{\nabla}_x H = 0 \tag{30}$$

From (18), (22) and (30) we have:

$$\tau_x(F) = \tau_x(G) = \tau_x(H) = 0 \tag{31}$$

From (21), because of $\overset{1}{\nabla}_x J = 0$ we have: $-J^2\tau_x = J\overset{21}{\nabla}_x J$ where J is F, G or H. If $J=F$, then we have: $\tau_x = F\overset{21}{\nabla}_x F$ and from (27) we get:

$$\tau_x = a(X)Id + b(X)F + c(X)G + d(X)H \tag{32}$$

Applying Theorem 1 to $\overset{1}{\Gamma}$ we obtain (28) and from (32) we get (29).

Conversely, if (28) and (29) are satisfied, then making use of (21) we obtain (27) which means that a pair $(\overset{1}{\Gamma}, \overset{2}{\Gamma})$ is compatible with a structure $Q = \{F, G, H\}$ on M.

REMARK 3. From (30) and (17) $\overset{m}{\Gamma}$ is a Q-connection and

$$\overset{m}{\nabla}_x F = \overset{m}{\nabla}_x G = \overset{m}{\nabla}_x H = 0 \tag{33}$$

COROLLARY 1. *Any linear connection* Γ *on a manifold* M *with an almost quasi-quater nion on structure* $Q = \{F, G, H\}$ *determines a pair* $(\overset{1}{\Gamma}, \overset{2}{\Gamma})$ *of linear connections given by* (28) *and* (29) *which is compatible with* Q.

Following [6] *we have the following*:

DEFINITION 4. [6]. For a pair of linear connections $\overset{1}{\Gamma}$ and $\overset{2}{\Gamma}$ on a manifold M the following operators:

$$\rho_{XY} = \frac{1}{2}\left([\overset{1}{\nabla}_x, \overset{2}{\nabla}_Y] + [\overset{2}{\nabla}_X, \overset{1}{\nabla}_Y] - \overset{1}{\nabla}_{[X,Y]} - \overset{2}{\nabla}_{[X,Y]}\right) \tag{34}$$

$$K_{XY} = \frac{1}{4}[\tau_X, \tau_Y] \tag{35}$$

are called operators of mixed and deformation curvatures respectively.

PROPOSITION 3. [6]. The following relations are satisfied:

$$2\rho_{XY} + 4K_{XY} = \overset{1}{R}_{XY} + \overset{2}{R}_{XY} \tag{36}$$

$$\rho_{XY} + K_{XY} = \overset{m}{R}_{XY} \tag{37}$$

where $\overset{1}{R}_{XY}$, $\overset{2}{R}_{XY}$ and $\overset{m}{R}_{XY}$ are curvature operators of connections $\overset{1}{\Gamma}$, $\overset{2}{\Gamma}$ and $\overset{m}{\Gamma}$ respectively.

Now, we give some properties of operators of mixed and deformation curvatures for a pair of connections compatible with an almost quasi-quaternion structure on M in the following:

THEOREM 3. *The operators of mixed and deformation curvatures of a pair of linear connections* $(\overset{1}{\Gamma}, \overset{2}{\Gamma})$ *which is compatible with an almost quasi-quaternion structure* $Q=\{F, G, H\}$ *on a manifold* M *satisfy the following conditions*:

(38) $$F\rho_{XY}=\rho_{XY}F,\quad G\rho_{XY}=\rho_{XY}G,\quad H\rho_{XY}=\rho_{XY}H$$

(39) $$FK_{XY}=K_{XY}F,\quad GK_{XY}=K_{XY}G,\quad HK_{XY}=K_{XY}H$$

(40) $$A\rho_{XY}=AK_{XY}=0,\quad B\rho_{XY}=\rho_{XY},\quad BK_{XY}=K_{XY}$$

where A, B *are given by* (5) *and* (6).

PROOF. *Making use of the fact that* $\overset{1}{\Gamma}$, $\overset{2}{\Gamma}$ *and* $\overset{m}{\Gamma}$ *are Q-connections*, $\overset{1}{R}_{XY}$, $\overset{2}{R}_{XY}$ *and* $\overset{m}{R}_{XY}$ *satisfy* (10), (11) *and* (12), *so from* (36) *and* (37) *we obtain* (38), (39) *and* (40).

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