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Z_p-ACTION ON SELF-DUAL CONNECTIONS

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1. Introduction

Let $G=Z_p$ where p is a prime number. Let M be a simply connected, closed, smooth 4-dimensional manifold with a positive definite intersection form, and a smooth G action on it. Let π: $E\to M$ be a quaternion line bundle with instanton number one and with G-action on E through bundle isomorphism such that the projection is a G-map. A connection on the vector bundle $E\to M$ is defined by a first order linear differential operator

$$\nabla: \Omega^0(E)\to\Omega^1(E).$$

The connection has a natural extension ∇_1: $\Omega^1(E)\to\Omega^2(E)$. The curvature R^∇ of the connection is defined as the composition $\nabla_1\cdot\nabla\in\Omega^2(\mathrm{End}E)$. The set $\mathscr{A}(E)$ of all connections on E has a natural affine structure. The set $\mathscr{G}$ of all bundle automorphisms on E forms a Lie group structure by fiberwise multiplication. The group $\mathscr{G}$ is called the gauge group of the bundle E. A connection ∇ is said to be self dual if $*R^\nabla=+R^\nabla$ where $*$ is the Hodge star operator on M. The gauge group $\mathscr{G}$ acts on connections and preserves the self duality. The group G acts on the connections as an extended gauge group and preserves the self-dual connections when we start with G-invariant metric on M. Then the moduli space $\mathscr{M}$ of the equivalence classes of self-dual connections on E is a G-space. This muduli space $\mathscr{M}$ may not be a manifold. By Uhlenbeck argument we may choose a G-invariant metric on M such that the fixed point set $\mathscr{M}^{*G}$ of the space $\mathscr{M}^*$ of irreducible self-dual connections is smooth.

In this paper we will compute the G-index of the fundamental elliptic complex for each G-invariant self-dual connections and we have Theorem 2.3. From this Theorem 2.3 we understand the local behavior of the group G in the moduli space $\mathscr{M}$. In section 3, we will study the G-action on the reducible self-dual connections and we have Theorem 3.1.

2. G-Index on Elliptic Complex

For a given $SU(2)$-vector bundle $E \to M$, let $P \to M$ be the associated principal $SU(2)$-bundle. Then we have the Lie algebra bundle $adp = PX_{SU(2)}^{su(2)}$ on M.

Assume that the fixed point set F of G-action on M is $F = \{P_i\}_{i=1}^{k_1} \cup \{T^{\lambda_i}\}_{i=1}^{k_2}$ where the $P_i's$ are isolated points and the T^{λ_i} are Riemann surfaces with genus λ_i.

Consider G-invariant elliptic complexi.e. the self-dual connection ∇ is a fixed point.

$$0 \longrightarrow \Omega^0(ad_c p) \underset{\delta_\nabla}{\overset{d^\nabla}{\rightleftarrows}} \Omega^1(ad_c p) \xrightarrow{d^\nabla_-} \Omega^2_-(ad_c p) \longrightarrow 0.$$

where δ^∇ is the formal adjoint operator of d^∇ and to get Banach spaces we give appropriate Sobolev norms on each term as usual.

Assume this complex is complexified. Then as usual we get an equivariant single Dirac operator:

$$D: \Gamma(V_+ \otimes V_- \otimes ad_c p) \longrightarrow \Gamma(V_- \otimes V_- \otimes ad_c p).$$

Let $K = \{e^{\frac{2\pi i}{p} n} : n = 0, 1 \cdots p-1\}$ be the character group of G.

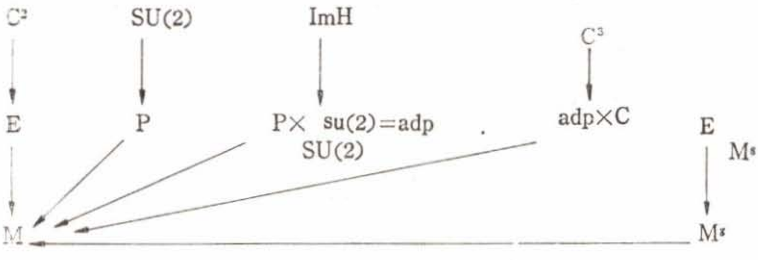
In the complex representations which is just the p-th roots of the unity. Let $g = e^{\frac{2\pi i}{p}}$ be a generator of G and let ∇ be an irreducible self-dual G-invariant connection. Then the analytic G-index of D is a virtual representation of G, namely $\mathrm{Index}_G(D) = H^1(ad_c p) - H^2(ad_c p) \in R(G)$. By Atiyah-Singer Fixed Point Theorem, we can compute the index

$$\mathrm{Index}_g(D) = \mathrm{trace}(g\, \mathrm{index}_G(D))$$

$$= (-1)^{\dim M^g} \frac{ch_g(V_+ - V_-)ch_g(V_-)ch_g(ad_c p)td(TM^g \otimes C)}{ch_g(\Lambda_{-1} N^g \otimes C)} [TM^g]$$

$$= (-1)^{\frac{\dim M^g}{2}} \frac{Ch_g(V_+ - V_-)ch_g(V_-)ch_g(ad_c p)td(TM^g \otimes C)}{e(TM^g)ch_g(\Lambda_{-1} N^g \otimes C)} [M^g]$$

To calculate $ch_g(ad_c p)$ let us examine Z_p action on $ad_c p$. Consider a diagram



To preserve the SU(2)-structure on $E|_{M^g}$, $g=e^{\frac{2\pi i}{p}}$ acts as

$$\begin{bmatrix} e^{\frac{2i}{p}k} & 0 \\ 0 & e^{\frac{2\pi i(p-k)}{p}} \end{bmatrix}$$

on $E|_{M^g}$ for some k and so on its associated principal bundle P. On the associated Lie algebra bundle $adp=P\times_{SU(2)}su(2)$, G acts by conjugation, i.e.

$$\begin{bmatrix} e^{\frac{2\pi ik}{p}} & 0 \\ 0 & e^{\frac{2\pi i(p-k)}{p}} \end{bmatrix}\begin{bmatrix} it & a \\ -\bar{a} & -it \end{bmatrix}\begin{bmatrix} e^{\frac{-2\pi ik}{p}} & 0 \\ 0 & e^{\frac{-2\pi i(p-k)}{p}} \end{bmatrix}=\begin{bmatrix} it & a \\ -\bar{a} & -it \end{bmatrix}$$

So G acts trivially on adp and $ad_c p=adp\otimes_R \mathbf{C}$. Thus in our case $ch_g(ad_c p)\equiv 3$.

The contribution to the $\text{Index}_g(D)$ ac an isolated fixed point $P_i\in F$. Let $\theta_1=\frac{2\pi r_i}{p}$, $\theta_2=\frac{2\pi s_i}{p}$ represent the representation of g on the normal bundle at P_i in M.

$$\frac{ch_g(V_+-V_-)ch_g(V_-)}{ch_g(\Lambda_- N^g\otimes C)}=\prod_{t=1}^{2}\frac{(e^{\frac{i\theta_t}{2}}-e^{\frac{-i\theta_t}{2}})e^{\frac{-i\theta_t}{2}}}{(1-e^{i\theta_t})(1-e^{-i\theta_t})}(e^{i\theta_1}+e^{i\theta_2})$$

$$=-\frac{1}{2}\left(1+\cot\frac{\pi r_i}{p}\cot\frac{\pi s_i}{p}\right)$$

Thus we have

LEMMA 2.1 $\text{Index}_g(D)|P_i=-\frac{3}{2}\left(1+\cot\frac{\pi r_i}{p}\cot\frac{\pi s_i}{p}\right)$

Next the contribution to the $\text{Index}_g(D)$ on a fixed point component $T^{\lambda_i}\subset F\subset M$ where T^{λ_i} is a Riemann surface with genus λ_i. Let g act on the normal bundle of T^{λ_i} in$M$ by $e^{\frac{2\pi i t_i}{p}}$ -multiplication on the fibers.

$$\frac{ch_g(V_+-V_-)ch_g(V_-)}{e(T^g)ch_f(\Lambda_{-1}N^g\otimes C)}$$

$$=\frac{(e^{\frac{x_1}{2}}-e^{\frac{-x_1}{2}})e^{\frac{-x_1}{2}}(e^{\frac{x_2}{2}}e^{\frac{\pi i t_i}{p}}-e^{\frac{-x_2}{2}}e^{\frac{-\pi i t_i}{p}})e^{\frac{-x_2}{2}}e^{\frac{-\pi i t_i}{p}}}{x_1(1-e^{x_2}e^{\frac{2\pi i t_i}{p}})(1-e^{-x_2}e^{\frac{-2\pi i t_i}{p}})}[e^{x_1}+e^{\frac{2\pi i t_i}{p}}e^{x_2}]$$

$$=\frac{e^{x_1}+e^{x_2}e^{\frac{2\pi i t_i}{p}}}{(1-e^{x_2}e^{\frac{2\pi i t_i}{p}})}=\frac{1+x_1+(1+x_2)e^{\frac{2\pi i t_i}{p}}}{1-(1+x_2)e^{\frac{2\pi i t_i}{p}}}=\frac{(1+e^{\frac{2\pi i t_i}{p}})+(x^1+e^{\frac{2\pi i}{p}}x_2)}{(1-e^{\frac{2\pi i t_i}{p}}-e^{\frac{2\pi i t_i}{p}}x_2)}$$

$$=\frac{1}{(1-e^{\frac{2\pi i t_i}{p}})}(1+e^{\frac{2\pi i t_i}{p}})+(x_1+e^{\frac{2\pi i t_i}{p}}x_2)\left[1+\frac{e^{\frac{2\pi i t_i}{p}}}{1-e^{\frac{2\pi i t_i}{p}}}x_2\right]$$

$$=\frac{1}{(1-e^{\frac{2\pi i t_i}{p}})}\left[x_1+e^{\frac{2\pi i t_i}{p}}x_2+\frac{(1+e^{\frac{2\pi i t_i}{p}})e^{\frac{2\pi i t_i}{p}}}{(1-e^{\frac{2\pi i t_i}{p}})}x_2\right]$$

$$=\frac{1}{(1-e^{\frac{2\pi i t_i}{p}})}x_1+\frac{2e^{\frac{2\pi i t_i}{p}}}{(1-e^{\frac{2\pi i t_i}{p}})^2}x_2$$

Here we only consider degree one part because, when we evaluate on the fundamental homology class $[T^{\lambda_i}]$, the other parts are all zero. x_1 and x_2 are the Euler classes of the tangent bundle and the normal bundle of T^{λ_i} in M respectively. We can calculate $x_2[T^{\lambda i}]=m_{T^{\lambda_i}}$ and $x_1(T^{\lambda_i})=2-2\lambda_i$

$$\text{Index}_g(D)|_{T^{\lambda_i}}=(-1)\frac{ch_g(V_+-V_-)ch_g(V_-)ch_g(ad_c p)td(T^g\otimes C)}{e(T^g)ch_{g(\Lambda_{-1}}N^g\otimes C)}[T^{\lambda_i}]$$

$$=-3\left[\frac{1}{(1-e^{\frac{2\pi i t_i}{p}})}x_1+\frac{2e^{\frac{2\pi i t_i}{p}}}{(1-e^{\frac{2\pi i t_i}{p}})^2}x_2\right][T^{\lambda_i}]$$

$$=-3\left[\frac{1}{(1-e^{\frac{2\pi i t_i}{p}})}+\frac{2e^{\frac{2\pi i t_i}{p}}}{(1-e^{\frac{2\pi i t_i}{p}})^2}T^{\lambda_i}\right]$$

$$=-6\left[\frac{(1-\lambda_i)+(m_{T^{\lambda_i}}+\lambda_i-1)e^{\frac{2\pi i t_i}{p}}}{(1-e^{\frac{2\pi i t_i}{p}})^2}\right]$$

Hence we have

LEMMA 2.2
$$\text{Index}_g(D)=\sum_{i=1}^{k_1}\text{Index}_g(D)|_{P_i}+\sum_{i=1}^{k_2}\text{Index}_g(D)|_{T^{\lambda_i}}$$
$$=\sum_{i=1}^{k_1}\left(-\frac{3}{2}\right)\left(1+\cot\frac{\pi r_i}{p}\cot\frac{\pi s_i}{p}\right)$$
$$+\sum_{j=1}^{k_2}(-6)\left[\frac{(1-\lambda_j)+(m_{T^{\lambda_j}}+\lambda_j-1)e^{\frac{2\pi i t_j}{p}}}{(1-e^{\frac{2\pi i t_j}{p}})^2}\right]$$

For $g^n\in z_p$ we can calculate the index

THEOREM 2.3.

$$\text{Index}_{g^n}(D)=\sum_{i=1}^{k_1}\left(-\frac{3}{2}\right)\left(1+\cot\frac{n\pi r_i}{p}\cot\frac{n\pi s_i}{p}\right)+\sum_{j=1}^{k_2}(-6)\left[\frac{(1-\lambda_j)+(m_{T^{\lambda_i}}+\lambda_j-1)e^{\frac{2n\pi i t_j}{p}}}{(1-e^{\frac{2n\pi i t_j}{p}})^2}\right]$$

where $n=1, \cdots p-1$ and r_i, s_i, t_i are determined by representations on the normal bundles of the fixed point set in M.

REMARK: (i) In the index calculation, $\text{td}(T^g\otimes C)=1+\frac{1}{2}c_1(T^g\otimes C)=1.$

(ii) Above $\text{Index}_g(D)$ is the topological index of D evaluated at g. If we know the exact data, namely the fixed point set, Z_p-representation nn the normal bundles and its Euler numbers then by the formula we calculate explicitly the topological index.

(iii) For example $G=Z_2$, $F=\{P, s^2\}\subset Cp^2$, $g=-1$.

$$\text{Index}_g(D)=\sum_{i=1}^{k_1}\left(-\frac{3}{2}\right)\left(1+\cot\frac{\pi r_i}{p}\cot\frac{\pi s_i}{p}\right)$$
$$+\sum_{j=1}^{k_1}(-6)\left[\frac{(1-\lambda_j)+(m_{T^{\lambda_j}}+\lambda_j-1)e^{\frac{2\pi i t_j}{p}}}{(1-e^{\frac{2\pi i t_j}{p}})^2}\right]$$
$$=\sum_{i=1}^{1}\left(-\frac{3}{2}(1+0)+\sum_{j=1}^{1}(-6)\frac{(1-0)+(1+0-1)(-1)}{[1-(-1)]^2}\right.$$

$= -3$

(iv) For simplicity these topological index

$$\mathrm{Index}_{g^n}(D) \overset{\text{put}}{\equiv} B_n \qquad n=0,\ 1,\ \cdots,\ p-1$$

From the G-invariant fundamental elliptic complex. Let us consider the G action on the virtual representation $H^1_{\nabla} - H^2_{\nabla} \in R(\dot{G})$ of the cohomology groups. Let us split H^1_{∇} and H^2_{∇} into the irreducible decompositions

$$H^1_{\nabla} = H^1_{g^0} \oplus H^1_{g^1} \oplus \cdots \oplus H^1_{g^{p-1}} \text{ and}$$

$$H^2_{\nabla} = H^2_{g^0} \oplus H^2_{g^1} \oplus \cdots \oplus H^2_{g^{p-1}} \text{ where } g = e^{\frac{2\pi i}{p}} \in G$$

acts on $H^j_{g^k}(j=1,\ 2)$ by the complex multiplication $e^{\frac{2\pi i}{p}k}$. Let $\dim_C H^1_{\nabla} = m$, $\dim_C H^1_{g^i} = m_i$, $\dim_C H^2_{\nabla} = n$ and $\dim_C H^2_{g^i} = n_i$. For each $g^i \in G$ we have the indices

LEMMA 2.4.

$$\begin{cases} \mathrm{Ind}_{g^0}(D) = (m_0 + m_1 + \cdots\cdots + m_{p-1}) - (n_0 + n_1 + \cdots + n_{p-1}) = B_0 = 5 \\ \mathrm{Ind}_{g^1}(D) = (m_0 + m_1 e^{\frac{2\pi i}{p}} + \cdots m + {}_{p-1} e^{\frac{2\pi i}{p}(p-1)}) - (n_0 + n_1 e^{\frac{2\pi i}{p}} + \cdots + n_{p-1} e^{\frac{2\pi i}{p}(p-1)}) = B_1 \\ \mathrm{Ind}_{g^2}(D) = (m_0 + m_1 e^{\frac{2\pi i}{p}2} + \cdots + m_{p-1} e^{\frac{2\pi i}{p}(p-2)}) - (n_0 + n_1 e^{\frac{2\pi i}{p}2} + \cdots + n_{p-1} e^{\frac{2\pi i}{p}(p-2)}) = B_2 \\ \quad \vdots \\ \mathrm{Ind}_{g^{p-1}}(D) = (m_0 + m_1 e^{\frac{2\pi i}{p}(p-1)} + \cdots + m_{p-1} e^{\frac{2\pi i}{p}}) - (n_0 + n_1 e^{\frac{2\pi i}{p}(p-1)} + \cdots + n_{p-1} e^{\frac{2\pi i}{p}}) = B_{p-} \end{cases}$$

Rearrange these equations to compute $(m_0 - n_0)$, $\cdots$ and $(m_{p-1} - n_{p-1})$

$$(2.5) \begin{cases} (m_0 - n_0) + (m_1 - n_1) + \cdots + (m_{p-1} - n_{p-1}) = B_0 \\ (m_0 - n_0) + (m_n - n_1) e^{\frac{2\pi i}{p}} + \cdots + (m_{p-1} - n_{p-1}) e^{\frac{2\pi i}{p}(p-1)} = B_1 \\ (m_0 - n_0) + (m_1 - n_1) e^{\frac{2\pi i}{p}2} + \cdots + (m_{p-1} - n_{p-1}) e^{\frac{2\pi i}{p}(p-2)} = B_2 \\ \quad \vdots \\ (m_0 - n_0) + (m_1 - n_1) e^{\frac{2\pi i}{p}(p-1)} + \cdots + (m_{p-1} - n_{p-1}) e^{\frac{2\pi i}{p}} = B_{p-1} \end{cases}$$

Using the fact $1 + e^{\frac{2\pi i}{p}} + \cdots + e^{\frac{2\pi i}{p}(p-1)} = 0$. Except the first row and the first column, each row and each column are permutations of the group G. For $0 < k < l \leq p-1$,

$$G\equiv\{1,\ e^{\frac{2\pi i}{p}k},\ e^{\frac{2\pi i}{p}2k},\ \cdots,\ e^{\frac{2\pi i}{p}(p-1)k}\}=\{1,\ e^{\frac{2\pi i}{p}(l-k)},\ e^{\frac{2\pi i}{p}2\cdot(l-k)},\ \cdots,\ e^{\frac{2\pi i}{p}(p-1)(l-k)}\}$$

By easy calculation we get

THEOREM 2.6

$$\begin{cases} m_0-b_0=\frac{1}{p}(B_0+B_1+B_2+\cdots+B_{p-1}) \\ m_1-n_1=\frac{1}{p}(B_0+B_1e^{\frac{2\pi i}{p}(p-1)}+B_2e^{\frac{2\pi i}{p}(p-2)}+\cdots+B_{p-1}e^{\frac{2\pi i}{p}}) \\ \vdots \\ m_{p-1}-n_{p-1}=\frac{1}{p}(B_0+B_1e^{\frac{2\pi i}{p}}+B_2e^{\frac{2\pi i}{p}2}+\cdots+B_{p-1}e^{\frac{2\pi i}{p}(p-1)}) \end{cases}$$

REMARK: (i) For a prime number p, $g=e^{\frac{2\pi i}{p}}\in G$, the fixed point set on M, $F=M^G=M^g=M^{g^2}=\cdots=M^{g^{p-1}}=\{P_i\}_{i=1}^{k_1}\cup\{T^{\lambda_i}\}_{i=1}^{k_2}$

(ii) the topological index $B_0=5$, B_1, $\cdots$, B_{p-1} is determined by the formula (2.3), and the virtual representatoon dimensions $m_i-n_i (i=0, \cdots, p-1)$ is determined by B_0, $\cdots$, B_{p-1} and the G-action on them.

(iii) For reducible self-dual connections we replace m_0+1 instead of m_0

(iv) $m_0-n_0=\frac{1}{p}(B_0+B_1+\cdots+B_{p-1})$ is the dimension of fixed point component containing ∇.

3. Z_p-Action on Reducible Self-Dual Connections

Next we would like to consider a Z_p action on reducible self-dual connections. Under our usual assumption on the bundle $E\to M$. We consider the space $H^2(R)$ of real valued harmonic 2-forms on M. Harmonic 2-forms ϕ means $d\phi=0=\delta\phi$ where $\delta=-*d*$. Since $*\Delta=\Delta*$, $H^2(R)=H^{2+}\oplus H^{2-}$. Every harmonic 2-form is self-dual since the intersection from is positive definite. Also self-dual connections are harmonic. For each reducible connection $\nabla=\nabla_1\oplus\overline{\nabla}_1$ on E, the assignment $\nabla\longmapsto\pm\Omega^{\nabla_1}$ gives a one-to-one correspondence between gauge-equivalence classes of reducible connections on E and pairs of closed 2-forms $\pm\Omega^{\nabla_1}$ with $\int_M\Omega^{\nabla_1}\wedge\Omega^{\nabla_1}=1$, where $\Omega^{\nabla_1}=\frac{-1}{2\pi i}R^{\nabla_1}$. Since M is simply connected, the de

Rham classes Ω^{∇_1} are uniquely determined by the integral classes in $H^2(M:Z)$. Also every integral class u with $u\cdot u=1$ comes from a reducible self-dual connection. Our manifold $M^4=CP^2\#\cdots\# CP^2$(n-copies).

Let $\{\pm b_1, \cdots, \pm b_n\}$ be a basis with $b_i b_j=\delta_{ij}$ in $H^2(M:Z)$. For any $g\in Z_p$, g is a diffeomorphism on M. So $\{\pm g^* b_1, \cdots, \pm g^* b_n\}$ is also a basis in $H^2(M:Z)$. In the moduli space $\mathcal{M}$ there are reducible self-dual connections $\{\nabla_1, \cdots, \nabla_n\}$ which corresponds $b_i=\frac{1}{2\pi i}R^{\nabla_i}$ for $i=1, \cdots, n$. Moreover $g^* b_i=g^*\frac{1}{2\pi i}R^{\nabla_i}=\frac{1}{2\pi i}gR^{\nabla_i}g^{-1}=\frac{1}{2\pi i}R^{g(\nabla_i)}$ which is the curvature from corresponding the connection $g^{(\nabla_i)}$. Since g is an isometry on M, $R^{g(\nabla_i)}$ is harmonic and the integral class $g^* b_i\cdot g^* b_i=g^*(b_i\cdot b_i)=g^*(I)=1$ where $b_i\cdot b_i$ is a generator in $H^4(M:Z)$. By definition this is the orientation class. Thus $g(\nabla_i)$ is a rducible self-dual connection.

THEOREM 3.1. *Z_p-action on $E\to M$ induces a action on the set of reducible connections in the moduli space $\mathcal{M}$ of the self-dual gauge equivalent connections. Moreover let $\check{\mathcal{M}}$ be the set of reducible connections in $\mathcal{M}$ and by setting $\nabla\longmapsto\frac{1}{2\pi i}R^{\Delta_1}$ where $\nabla=\nabla_1+\overline{\nabla}_1$, the diagram*

$$\begin{array}{ccc} H^2(M:Z) & \xrightarrow{\ g\ } & H^2(M, Z) \\ \uparrow{\scriptstyle \Omega} & & \uparrow{\scriptstyle \Omega} \\ \check{\mathcal{M}} & \xrightarrow{\ g\ } & \check{\mathcal{M}} \end{array} \text{ commutes.}$$

REMARK. (i) Suppose that ∇ is a reducible self-dual connection. Then the isotropy group of ∇, $\Gamma^{\nabla}=\{g\in\mathcal{G}\mid g\nabla g^{-1}=\nabla\}=S^1$. For any $h\in Z_p$, $\Gamma^{h(\nabla)}=\{hgh^{-1}\in\mathcal{G}\mid g\in\Gamma^{\nabla}\}=S^1$ because $(hgh^{-1})\cdot h(\nabla)=hgh^{-1}$. $h\nabla h^{-1}\cdot hg^{-1}h^{-1}=hg\nabla g^{-1}h^{-1}=h(\nabla)$. Also $hg_1h^{-1}=hg_2h^{-1}$ implies $g_1=g_2$.

Since Z_p preserves the self-duality, $h(\nabla)$ is also a self-dual reducible connection.

(ii) On the Z_p-action over $E\to M^4$ and $h\in Z_p$. Let $b_1\in H^2(M:Z)$ with $b_1\cdot b_1=1$, then there is a complex line bundle $L_{b_1}\to M$ with its Euler class b_1. And consider the map

$$\begin{array}{ccc} h^{*}L_{b} & \longleftarrow & L_{b_1} \\ \downarrow & & \downarrow \\ M & \xrightarrow{h} & M \end{array}, \qquad \begin{array}{ccc} h^{*}: H^{2}(M:Z) & \longrightarrow & H^{2}(M:Z) \\ b_1 & \longmapsto & h^{*}(b_1) \end{array}$$

The induced bundle $h^{*}(L_{b_1})$ over M is exactly the bundle corresponding reducible self-dual connections ∇ and $h(\nabla)$

$$\Gamma^{\nabla} = \{g \in \mathcal{G} \mid g(\nabla) = \nabla\} = S^1 \subset \mathcal{G}$$
$$\Gamma^{h(\nabla)} = \{hgh^{-1} \mid g \in \Gamma^{\nabla}\} = S^1 \subset \mathcal{G} \text{ and}$$

$g = \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} \in \Gamma^{\nabla}$. From the bundle splitting, the following diagram commutes and preserves the splittings.

$$\begin{array}{ccc} E = L_{b_1} \oplus \overline{L}_{b_1} & \xrightarrow{h} & L_{h(b_1)} \oplus \overline{L}_{h(b_1)} \\ \downarrow g & & \downarrow hgh^{-1} \\ L_{b_1} \oplus \overline{L}_{b_1} & \xrightarrow{h} & L_{h(b_1)} \oplus \overline{L}_{h(b_1)}. \end{array}$$

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