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## EXISTENCE OF POLYNOMIAL INTEGRATING FACTORS

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Abstract: We study existence of polynomial integrating factors and solutions F(x, y) = c of first order nonlinear differential equations. We characterize the homogeneous case, and give algorithms for finding existence of and a basis for polynomial solutions of linear difference and differential equations and rational solutions or linear differential equations with polynomial coefficients. We relate singularities to nature of the solution.

Solution of differential equations in closed form to some degree might be called more an art than a science: The investigator can try a number of methods and for a number of classes of equations these methods always work. In particular integrating factors are tricky to find.

An analogous but simpler situation exists for integrating inclosed form, where for instance there exists a criterion for when an exponential integral can be found in closed form.

In this paper we make a beginning in several directions on these problems, for 2 variable ordinary differential equations. The case of exact differentials reduces immediately to quadrature. The next step is perhaps that of a polynomial integrating factor, our main study.

Here we are able to provide necessary conditions based on related homogeneous equations which probably suffice to decide existence in most cases.

As part of our investigations we provide complete algorithms for existence of and finding a basis for polynomial solutions of linear differential and difference equations with polynomial coefficients, also rational solutions for such differential equations.

Our goal would be a method for decidability of whether any differential equation Mdx+Mdy=0 with polynomial M, N has algebraic solutions(or an undecidability proof). We reduce the question of all solutions algebraic to singularities but have not yet found a definite procedure to find their type.

We begin with general results on the set of all polynomial solutions and integrating factors.

Consider a differential equation Mdx + Ndy where M, N are nonreal polynomials

in x, y with no common factor. When does there exist an integrating factor u which is (i) polynomial (ii) rational? In case (i) the solution F(x, y) = c will be a polynomial. We assume all functions here are complex analytic polynomial in some open set.

PROPOSITION 1. Let f(x, y) be a function g(h(x, y)), where g is non constant, in some open set. Then h(x, y) = c is also a solution Moreover if h(x, y) is the polynomial of minimum degree giving a solution then any polynomial solution f has this form, with g polynomial.

For the second statement, let f=c be a nonconstant polynomial solution and h=c the polynomial solution of minimum degree. Then h, f are solutions for all complex numbers because of the polynomial property. There exists a function g defined at least locally by

$$g'(h(x, y)) = \frac{f_x}{h_x} = \frac{f_y}{h_y}$$

and we have g is algebraic. Suppose first g is not rational. Let F(x, y) be the field of rational functions of x, y over c.

Let U be the subfield of all functions which are rational in f, h. We will show U is isomorphic to a subfield of purely transcendental degree 1 extension. By Luroth's theorem it itself will be such.

There exists  $y_0$  such that setting  $y=y_0$  gives an isomorphism on the field of rational functions in h(x, y), f(x, y). Take  $y_0 \in c$  transcendental over the field of coefficients of f, h. No algebraic function of  $f(x, y_0)$  is locally constant except a constant rational function. Therefore this is well defines and a field homomorphism. Therefore it is 1-1.

By Luroth's Theorem there exists a rational function r(x, v) such that both h(x, y), f(x, y) are rational functions of r(x, y). So some rational function of r(x, y) is a polynomial

$$\frac{P(r(x,y))}{S(r(x,y))}$$

Write this as

$$\frac{h_1(n(x, y), d(x, y))}{h_2(n(x, y), d(x, y))}$$

where  $h_i$  are homogeneous, n, d numerator and denominator of r in lowest terms.

Then since P, S are relatively prime, so are  $h_1$ ,  $h_2$ . By the Euclidean algorithm polynomials  $p_1, p_2$  exist with  $p_1h_1+p_2h_2=d$  (looking at the ring of function polynomial in n/d). But  $h_1/h_2$ . We have then that h is a power of d. Similar reasoning gives  $h_2$  a power of n. So  $h_2$  is constant.

Write *h* as a product  $\prod (a_i n + b_i d)$ . Then some  $a_i n + b_i d$  is constant and the rest are multiples of this. It follows that a linear fractional transformation

$$r \longrightarrow \frac{r}{ar+b}$$

is a polynomial which also generates the field. It must coincide then with h. This last statement also applies with h, f rational.

PROPOSITION 2. Any polynomial integrating factor is a polynomial in h times the integrating factor giving h.

PROOF. The solution will be a polynomial f(h). Its derivatives will be f'(h) $h_x, f'(h)h_y$ . The divisor, f'(h) times the g.c.d. of  $h_x, h_y$  will be the integrating factor.

The homogeneous case has known rational integrating factor

$$\frac{1}{Mx+Ny}$$

but has general importance because of the terms of highest(or lowest) degree.

THEOREM 1. For a nonconstant polynomial integrating factor to exist, h must be divisible by V for some polynomial V of degree at least 1. Conversely if h is a homogeneous polynomial having a square factor, a nonconstant polynomial integrating factor exists.

PROOF. The second statement is immediate. Suppose h is the least degree solution of Mdx+Ndy, M, N homogeneous degree r, u an integrating factor. The top degree terms of U give an integrating factor, so we can assume u, and therefore h, are homogeneous. Assume the degree of h is minimal for this. Let h have degree k, h=xf(z,1) where z=y/x, u=xu(z,1). Then u/h,  $h_y$ . It follows u(z,1)|f(y,1), f'(z,1). So f(z,1) has u(z,1) as a square factor. Likewise  $f\left(1, \frac{x}{y}\right)$  has  $u\left(1, \frac{x}{y}\right)$  has a square factor. The ratio must be a polynomial divided by x in the first case, y in the second so it must be a polynomial.

PROPOSITION 3. For M, N homogeneous of equal degree, a rational solution exists if and only if some multiple of numbers  $A_r$  are rational in the case of no multiple roots of Mx+Ny. Here let r be the roots of My+Mx in  $z=\frac{x}{y}$ including an infinite root if y|n, and  $A_r$  are for finite roots, assuming M, N are relative prime

$$A_{r} = \frac{M(r,1)}{M(r,1) + M_{x}(r,1)r + N_{x}(r,1)}$$

and for infinite

$$A_{\infty} = \frac{N(1,0)}{M(1,0) + N(1,0)}$$

then the solution is

$$\prod_{r} (y - rx)^{A_r} \cdot y^{A_{\infty}} = c$$

PROOF. The solution is obtained from

$$\frac{M(x,y)dx}{Ny+Mx}, \frac{N(x,y)dy}{Ny+Mx}$$

by deleting repeated terms. We expand each in partial fractions where  $Ny+Mx=\pi(x-ry)y^{\varepsilon}$  over finite roots, where  $\varepsilon=0$  if no infinite roots occur, else=1. Let

$$\frac{M(x,y)}{Mx+Ny} = \frac{A_i}{x-r_iy} + \frac{B}{y}$$

then the solution is  $A\log(x-r y)+(y)$  where considering

$$\frac{N(x,y)}{Mx+Ny} = \Sigma \frac{B_i}{x-r_i y} + \frac{A}{y}$$

we obtained the term  $A\log y$ . This proves the solution formula and that implies the rationality condition. It remains to check the A.

$$\frac{M(x,y)}{Mx+Ny} = \Sigma \frac{A_i}{x-r_i y} + \frac{B}{y}, y=1$$
$$\frac{M(x,1)}{Mx+N} = \Sigma \frac{A_i}{x-r_i} + \frac{B}{1}, \text{ let } x \rightarrow r$$

Then asymptotically the two sides are

$$\frac{M(r_i, 1)}{D_x(Mx+N)(x-r_i)} \text{ and } \frac{A_i}{x-r_i}. \text{ So}$$
$$A_i = \frac{M(r_i, 1)}{M(r_i, 1) + rM(r_i, 1) + N(r_i, 1)}$$

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To get  $A_{\infty}$ , we study

$$\frac{N}{Mx + Ny} = \Sigma \frac{B_i}{x - r_i y} + \frac{A_\infty}{y}$$

Asymptotically as  $y \rightarrow 0$ , x=1

$$\frac{N(1,0)}{yD(M+Ny)} = \frac{N(1,0)}{y(M_y(1,y)+yN'(1,y)+N(1,y))} = \frac{A_{\infty}}{y}$$
$$A_{\infty} = \frac{N(1,0)}{N(1,0)+M_y(1,0)}$$

LEMMA 1. If U(x, y), V(x, y) are products of powers of (x-ay) and any  $g(U(x, y)+\log V(x, y))$  is an algebraic function then U is a function of V or V is constant.

PROOF. If not, take a curve along which V(x, y) is constant. Then g must be algebraic on this curve. So it is algebraic. But then take a curve along which U(x, y) is constant. We have nonalgebraic.

THEOREM 2. Let M, N be homogeneous of equal degree, relatively prime, Mx+Ny having a multiple root. Expand

$$\frac{M}{Mx+Ny}$$
 as  $\Sigma \frac{A_{ir}}{(x-ry)^{i}} + p(y)$ 

and

$$\frac{N}{Mx+Ny}$$

as

$$\frac{A_{i\infty}r}{y^i} + \frac{B_{ir}}{(x-r y)^i} + Q(x)$$

Then a rational solution exists if

- (1) There is only one root r of Mx+Ny
- (2) All A<sub>ir</sub> are zero, or

(3) 
$$\sum_{i>2} \frac{A_{ir}}{r(x-ry)^{i-1}} \text{ is a function of }_{i=1} \Pi (x-ry)^{A_{ir}}$$

In case(1) x-ry=c and in cases (2), (3)

$$\sum_{r>2} \frac{A_{ir}}{i(x-r y)^{i-1}} = c$$

is a solution. In all other cases there is no rational solution f(x, y) = c.

PROOF. We have a partial fraction expansion

$$\int \frac{M}{Mx + Ny} = \Sigma \int \frac{A_{ir}}{(x - ry)^i} + Q(y) = \int \frac{N}{Mx + Ny} + P(x)$$

giving the solution. So it is

$$\sum_{i>2} \frac{A_{ir}}{(x-r y)^{i-1}} + \sum_{i=1}^{n} A_{ir} \log(x-ry)$$

A function of this is rational if there is one and only one root (take x-ry) or if all A 1 are zero. It also happens if

$$\sum_{i>2} \frac{A_{ir}}{(x-ry)^{i-1}}$$

is a function of

$$\prod_{i=1}^{n} (x - ry)^{A^{ir}}$$

If the one is not a function of the other and both are nonzero, then the sum cannot be rational by Lemma 1.

LEMMA 2. Let f be a (Laurent) power series in z. Then f represents a rational function of z for sufficiently small z=0 if and only if for all large n the coefficient of  $x^n$  can be expressed as a finite sum of series of the form  $f(n)r^n$  where f is a polynomial, r is a complex number. Here r is a root of the denonimator, the degree of f in its multiplicity minus 1.

PROOF. Any finite set of terms can be altered by adding *a* sum of powers of *z*. Then we can have a rational function by expressing the powers as a sum of  $a_i(1-rz)^{-1} = \sum a_i r^i z^i \binom{i+r-1}{i}$  where

$$\frac{i.i+1\cdots r+r-1}{1.2\cdots i}\binom{i+r-1}{i}$$

Conversely any rational function by first dividing and taking a remainder can be expressed as

$$\frac{P(x)}{R(x)}$$
, deg  $P(x) < \deg R(x)$ 

Then by expressing R(x) in terms of partial fractions we get the form described. Given a differential equation polynomial in x is there an algorithm to tell if a rational solution exists?

PROPOSITION 4. If f is a polynomial solution of Mdx+Ndy then the highest degree term of f forms a polynomial solution Mdx+Ndy where M, N are the homogeneous polynomial, consisting of the highest degree terms of M, N. The same holds for lowest degree terms.

PROOF. Let U be an integrating factor which will be polynomial

$$\left(UM = \frac{\partial f}{\partial x}, UN = \frac{\partial f}{\partial y}\right)$$

then the highest terms of UM equal the highest degree terms of  $\frac{\partial f}{\partial x}$ , and the same for UN,  $\frac{\partial f}{\partial x}$ . But those came from the highest degree terms of U, M, N, f respectively. S if we denote them by subscript 0,  $U_0M_0 = \frac{\partial f_0}{\partial x}$ ,  $U_0N_0 = \frac{\partial f_0}{\partial y}$ . The same holds for lowest degree.

Suppose we have chosen a particular solution (nonunique) for the highest (or lowest) degree terms. One way to proceed is to look next at the second highest or lowest degree terms. This will also be homogeneous and thus we will have a linear equation which can be translated into a first order nonhomogeneous linear equation in one variable,

$$p_n y^{(n)} + p_1 y^{(n-1)} + \dots + p_0 = 0$$

There are linear equations relating the coefficient of x to coefficients of lower degrees. The least degree that this produces in the equation is  $\min(n_i - i + m)$  where  $c_i x^{n_i}$  is the minimum degree nonzero term of P. This term is

$$\sum_{s} c_{i} \frac{m!}{(m-(n-i))} x^{n_{i}-i+m}$$

since we have  $c_i x^{n_i}$  times  $D^i x^m$ . Here S is the set of *i* such that  $n_i - i + m$  equals the minimum. What is the lowest degree coefficient which can produce a term of degree  $n_i - i + m$ ? The coefficient of  $x^r$  can produce a term of degree at most  $r - i + \deg P_i$ . So this gives a definite bound.

PROPOSITION 5. Let  $p_0 y^{(n)} + \dots + p_n y = p_{n+1}$ , be any linear differential equation with coefficients polynomial in x. There exists an algorithm for deciding existence of and finding a polynomial solution.

PROOF. Change x to x-c so that  $p_0$  has nonzero constant terms. The previous

discussion shows we have a linear set of equations for the coefficients  $c_m$  in of  $x^m f_0(m)c_m + f(m)c_{m-1} + \dots + f_k(m)c_{n-k} = 0$ , for some k, where f(m) are polynomial in m.  $f_0$  has degree n and all their  $c_m$  have degree at most n, i, e, come from expressions in nth derivative. These equations are necessary and sufficient for a power series solution. We can interpret them as a matrix expressing  $c_m$ ,  $c_{m-1}$ ,  $c_{m-k+1}$  in terms of  $c_{m-1}, c_{m-2} \cdots c_{m-k}$ . Write  $c(m) = (c_m, c_{m-1} \cdots c_{m-k+1})$ . Then c(m) = c(m-1) A(m) where

$$A(m) \text{ is } \begin{vmatrix} -f_1 & -f_2 \\ f_0 & f_0 \\ 1 & 0 & \cdots \\ 0 & 1 & \cdots \\ 0 & 1 & \cdots \\ 0 & 0 \end{vmatrix}$$

If there exists a solution of degree m then $(c_{m+k-1}\cdots, c_m) \neq 0$  but  $(c_{m+k}\cdots, c_{m+1})=0$ . So this matrix for m+k is singular. Here the first row is  $\frac{-f_1}{f_0} \frac{-f_2}{f_0} \cdots \frac{-f_k}{f_0}$ , otherwise rows have only one 1. The determinant is  $\frac{-f_k}{f_0}$  which will have only a finite number of roots (assuming k chosen so  $f_k \neq 0$ ) then we have an upper bound on the degree m, since f(m+k)=0. Then we just have simultaneous linear equations on the coefficients.

THEOREM 2. If  $\Sigma f(n)x$  is any power series solution of the linear equation

$$\sum_{i=1}^{k} P(x) y^{(i)} = R(x)$$

$$P_{i}(x), R(x) \text{ polynomials}$$

then for all  $n \ge \deg R(x) + 2k$  we must have  $[f(n), f(n-1)], \cdots f(n-k+1] = [f(n-1), \cdots f(n-k)] A$  where A is the matrix

$$\begin{vmatrix} \frac{-f_1}{f_0} & \frac{-f_2}{f_0} & \cdots & \frac{-f_k}{f_0} \\ 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \end{vmatrix}$$

of the last theorem. Here f, are determined polynomials in n.

This follows from the proof of the last theorem.

THEOREM 3. Let  $p_n(x) \Delta^n y + p_{n-1}(x) \Delta^{n-1} y + \cdots + p(x) y = p_{n+1}(x)$  be linear differencial equation with polynomial coefficients. There exists an algorithm to decide

if it has a polynomial solution.

PROOF. We will expand y as an infinite series  $\Sigma c_n \begin{pmatrix} x \\ n \end{pmatrix}$  with  $\Delta \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ n-1 \end{pmatrix}$ . Write  $p_j(x)$  first as similar(finite sum), then expand it in the form  $\Sigma \begin{pmatrix} x-m \\ k \end{pmatrix}$   $d_{kj}(m)$  for certain polynomials  $d_{ij}$ , m a variable.

We will use the identity

$$\binom{x}{n+k} = \binom{x}{n} \frac{(x-n)\cdots(x-n-k-1)}{n+1\cdots(n+k)}$$
$$= \frac{\binom{n}{n}\binom{x-n}{k}}{\binom{n+k}{k}}$$

Now we look at the coefficient of  $\binom{x}{n}$  in the equation, for *n* large. The term  $\Delta^{j}(c_{i}\binom{x}{i}) = c_{i}(i-j)$  produces a term.

$$c_{i} \binom{x}{i-j} \binom{x-i+j}{n-i+j} d_{n-i+j} j^{(i-j)} \binom{n}{n-i+j} / \binom{n}{n-i+j}$$
  
=  $c_{i} \binom{x}{n} d_{n-i+j} j^{(i-j)} \binom{n}{n-i+j}$ 

Each i, j produces at most one such term. The equation then reduces to  $\Sigma c_i$  $d_{n-i+j}j^{(i-j)}(n-i+j)=0$ . Here j=1 to n, and i is such that  $n-i+j\geq 0$  and  $n-i+j\leq \deg(P_i)$ . Let k=n-i+j. This gives an equation

$$\sum_{k=0}^{\max^{des}p_j}\sum_{j=1}^n c_{n-k+j}d_{kj}(n-k)\binom{n}{k}$$

as before this gives a polynomial recursion  $f_s(n)c_{n+s} = f_{s-1}(n) c_{n+s-1} \cdots f_0(n)c_m$ . As before we write in matrix form. For the highest coefficient the matrix must be singular. There are only a finite number of numbers n for which this happens. So this bounds the degree of the polynomial and the rest is simultaneous linear equations.

LEMMA 3. The function  $a^n$  are linearly independent over rational functions of n, for distinct  $a \neq 0$ .

PROOF. If not then by taking a sum  $\Sigma a_s^n x^n f_i(n) = 0$ , some sum  $\Sigma \frac{c_{is}}{(x-a)_i^s} = 0$ .  $c_{is} = 0$ . This is false.

Cor. For any linear differential equation with polynomial coefficients there exists an algorithm to decide if a rational solution exists.

PROOF. By Theorem 2 it is enough to tell whether an equation v(n+1) = Mv(n) has a solution where  $v(n+1) = [f(n+k), \cdots f(n+1)]$  and f(n) is a sum of terms  $f_i(n)r_i^m$ ,  $f_i$  polynomials. We can limit the  $r_i$  to a finite set of numbers since if f=r/s is a rational solution in lowest terms then any prime factor S must divide the coefficient of the highest derivative (expand f in partial fractions, otherwise the denominator of this term has a power of  $x-r_i$  which cannot be cancelled) If this equation holds for all n it holds for the coefficients  $f_i(n)$  of the powers of each  $r_i$  by the lemma. This gives a finite difference equation for  $f_i$ .

By the theorem on difference equations they can be solved. This bounds the degree of denominators. By multiplying by the largest possible denominator we reduce to finding polynomial solutions.

THEOREM 4. The following are equivalent, for a differential equation Mdx + Ndy=0. Where M, N are relatively polynomials in x, y: (1) all solution in any open disk are algebraic (2) all solutions, if extended as far as possible have singularities only poles and branch points of finite order (3) there is a general solution a(x, y)=c an algebraic c an arbitrary parameter (4) there are algebraic solutions for some  $x_0$ , through all  $y_0$  in a cantor set such that N(x, y)=0.

PROOF. Whenever  $N \neq 0$  there exists a solution locally in a disk. If N=0 identically then also M=0 identically which will not happen for N, M relatively prime. When N=0 we can write a local solution uniquely as an analytic function in a parameter, the initial y value  $y_0$  at  $x_0$ . The set S where the solution exists will be open in c and those where it has degree at most n will be closed subset. By Baire category for some  $\alpha$  we have it is somewhere dense within each open subset. But if we have an open subset where the degree is at most y we can write the coefficients as analytic functions or the parameter locally. All derivatives of a are rational in y at x. This gives simultaneous linear equations in the coefficients and proves they are rational in y. This solution is then valid for a dense open set not just a disk. It also means y is an algebraic function a(x, y)=c. So (4) implies(3).

Suppose (3) holds. Then this solution is valid for all c and then exists a local solution(possibly multivalued) near any point. Therefore all solutions where N=0 must coincide with part of each a solution. But every solution has

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N=0 almost everywhere, as above so all solutions fit this form (with a maximally extended). Equivalence of (1), (2) is a well known theorem and (1)=(4). (see Ahifors [1]).

THEOREM 5. Suppose an equation Mdx+Ndyo, M, N polynomial has a general solution a(x,y)=c (a(x,y) algebraic). Then when we make fractional linear substitutions for x, y, there exists  $n \in Z$  such that at all but a finite number of points ( $x_0, y_0$ ) of the singular curve N(x,y)=0 there exists a local power series solution

$$y = \sum a_m z^m$$

in  $z=(x-x_0)^{1/n}$ , or a solution x=c.

PROOF. Ahlfors proves at any branch point of finite order there exists such a power series representation. It suffices to show almost all (x, y) on N are branch points of a solution of bounded order. Some $(x_0, y_0)$  can be outside any solution. e.g., take as equation h(x, y, c)=1 where h(0, u, c)=0. If we wrote a(x, y)=c in the form h(x, y, c)=0 (irreducible) it is sufficient in order for a solution to exist in a neighborhood of  $x_0, y_0$  that not all coefficients of  $c^n$  be zero, n>0. If not each coefficient is divisible by some factor of N. Suppose infinity many points  $x_0$  exists. Then we can find an irreducible algebraic curve containing infinitely many points, so on which the coefficients vanish. So we may take all coefficients divisible by an irreducible factor  $N_0$  of N. Now assume that for a general algebraic nontrivial solution h(x, y, c) rational in x, y, polynomial in c, the degree in c is minimal. Let the constant term be 1.

If the degree is greater than 1, we can reduce it as follows: differentiate with respect to x then replace y' by M/N, divide by b a power or n. In this process everything goes to parts divisible by the factor  $N_0$  of N except for  $N_{0y}y'$  which goes to  $N_{0y}M$ . Then  $N_0$  and N must have a common factor. This cannot happen for N irreducible.

Next suppose that the polynomial has degree 1. We may take a general solution as a(x, y) = c, a(x, y) rational. Such a solution will have the property mentioned. All  $x_0, y_0$  will be on a solution, and if a singularity occurs it will not branch type degrees of singularities are bounded.

## CONCLUSION

We have a number of algorithmically decidable necessary conditions for existence of a polynomial integrating factor. At any point  $x_0, y_0$ , we can replace x, y by  $x-x_0, y-y_0$ . The terms of highest and lowest degree must themselves have integrating factors. If we write homogeneous polynomials as functions of 1 variable only we can extend this condition up and down finitely many degrees by taking an ordinary equation system in the polynomials.

We raise these questions: (1) How likely are conditions of this type to be sufficient? (2) Can we bound the degree of an algebraic solution? (3) If a(x, y) = c is an algebraic solution, M, N polynomials must g(a) be rational for some g?

## REFERENCES

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