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NUMERICAL SOLUTION OF A GENERAL CAUCHY PROBLEM

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Abstract: In this work, two numerical schemes are proposed for solving a general form of Cauchy problem. Here, the problem, to be defined, consists of a system of Volterra integro-differential equations. Picard's and Seidel'a methods of successive approximations are ued to obtain the approximate solution. The convergence of these approximations is established and the rate of convergence is estimated in every case.

1. Introduction

Consider in particular the first-order equations

$$\left.\begin{aligned} &\frac{dy_i(t)}{dt}=f_i[t,\ y_1,\ y_2,\ \cdots,\ y_n;\ \int_{t_i}^{t} g_i(t,\ s,\ y_1,\ y_2,\ \cdots,\ y_n)ds], \\ &\text{with the conditions} \\ &y_i(t_i)=\alpha_i(0\leq t_1\leq t_2\leq\cdots\leq t_n\leq T,\ 0\leq t\leq T,\ i=\overline{1,\ n}), \end{aligned}\right\} \quad (1)$$

where α_i are constants and the functions $f_i(t,\ y_1,\ y_2,\ \cdots,\ y_n,\ Z(t))$. $g_i(t,\ s,\ y_1,\ y_2,\ \cdots,\ y_n)$, $(i=\overline{1,\ n})$ are defined in the domains:

$$D_1(of\ f_i)=[0,\ T]\times[\alpha_1-R,\ \alpha_1+R]\times[\alpha_2-R,\ \alpha_2+R]\times\cdots\times \times[\alpha_n-R,\ \alpha_n+R]\times[-R,\ R],$$

$$D_2(of\ g_i)=[0,\ T]\times[0,\ T]\times[\alpha_1-R,\ \alpha_1+R]\times\cdots\times[\alpha_n-R,\ \alpha_n+R].$$

Cauchy problem (1) arises in the equilibrium situations that take place in studying thin elastic shells of revolution and in nuclear collision problems [6], [4].

Section 2 is concerned with the application of Picard's method to prove the existence and uniqueness of the solution of Cauchy problem (1). The acquired results are given in theorem 1. Section 3 deals with the construction of Seidel successive approximations (since methods using the most up-to-date information tend to be better than those using older information) for solving the considered

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problem numerically, and the relevant theorem is proved.

2. Picard's method of successive approximations

Picard's method when applied to system (1) gives:

$$\left.\begin{aligned} &y_i^{(0)}(t)\ (\text{the initial approximation}) = \alpha_i,\ (0 \le t \le T),\ y_i^{(k+1)}(t_i) = \alpha_i, \\ &\frac{dy_i^{(k+1)}(t)}{dt} = f_i[t,\ y_1^{(k)},\ y_2^{(k)},\ \cdots,\ y_n^{(k)};\ \int_{t_i}^{t} g_i(t,\ s,\ y_1^{(k)},\ y_2^{(k)}, \\ &\qquad\qquad \cdots,\ y_n^{(k)})ds],\ k=0,\ 1,\ 2,\ \cdots \end{aligned}\right\} \tag{2}$$

THE SUFFICIENT CONDITIONS FOR CONVERGENCE 2.1. Assume that the two sets of functions $f_i(t,\ y_1,\ y_2,\ \cdots,\ y_n,\ Z)$ and $g_i(t,\ s,\ y_1,\ y_2,\ \cdots,\ y_n)$ satisfy the following conditions:

a) they are continuous and bounded for any $(t,\ y_1,\ y_2,\ \cdots,\ y_n,\ Z(t)) \in D_1$ and $(t,\ s,\ y_1,\ y_2,\ \cdots,\ y_n) \in D_2$, i.e.

$$|f_i(t,\ y_1,\ y_2,\ \cdots,\ y_n,\ Z(t))| \le M_1,\ M_1 = \text{const.},$$
$$|g_i(t,\ s,\ y_1,\ y_2,\ \cdots,\ y_n)| \le M_2,\ M_2 = \text{const.},$$

b) for arbitrary $(t,\ y_1,\ y_2,\ \cdots y_n,\ Z_1(t)),\ (t,\ \bar{y}_1,\ \bar{y}_2,\ \cdots,\ \bar{y}_n,\ Z_2(t)) \in D_1$ and $(t,\ s,\ y_1,\ y_2,\ \cdots,\ y_n),\ (t,\ s,\ \bar{y}_1,\ \bar{y}_2,\ \cdots,\ \bar{y}_n) \in D_2$, they satisfy Cauchy-Lipschitz condition, i.e.

$$|f_i(t,\ y_1,\ y_2 \cdots,\ y_n,\ Z_1) - f_i(t,\ \bar{y}_1,\ \bar{y}_2,\ \cdots,\ \bar{y}_n,\ Z_2)|$$
$$\le L_1 [\max_i |y_i - \bar{y}_i| + |Z_1 - Z_2|],\ L_1 = \text{Const.},$$
$$|g_i(t,\ s,\ y_1,\ y_2,\ \cdots,\ y_n) - g_i(t,\ s,\ \bar{y}_1,\ \bar{y}_2,\ \cdots,\ \bar{y}_n)|$$
$$\le L_2 \max_i |y_i - \bar{y}_i|,\ L_2 = \text{const.}$$

c) if $M = \max\{M_1,\ M_2,\ L_1,\ L_2\}$, then $MT \le R$.

Using mathematical induction and conditions a), b), c) it can be shown that

$$|y_i^{(k)}(t) - \alpha_i| \le R,\ (i=1,\ n;\ k=0,\ 1,\ 2,\ \cdots).$$

We shall prove that the sequence of functions $\{y_i^{(k)}(t)\}$ $k=0,\ 1,\ \cdots$, under the conditions a), b) and c) converges uniformly (i.e. a Cauchy sequence). In fact for $0 \le t \le T$, $1 \le i \le n$, we have

$$|y_i^{(1)}(t) - y_i^{(0)}(t)| = |y_i^{(1)}(t) - \alpha_i|$$

$$=\left|\int_{t_i}^{t} f_i[s,\ y_1^{(0)},\ y_2^{(0)},\ \cdots,\ y_n^{(0)};\ \int_{t_i}^{s} g_i(s,\ \tau,\ y_1^{(0)},\ y_2^{(0)},\ \cdots,\ y_n^{(0)})d\tau]\,ds\right|$$

$$\leq M_1\left|\int_{t_i}^{t} ds\right|=M_1|t-t_i|,$$

hence for $k=1,\ 2,\ 3,\ \cdots,$ we get

$$|y_i^{(k)}(t)-y_i^{(k-1)}(t)|\leq L_1\left|\int_{t_i}^{t}\left\{\max_i|y_i^{(k-1)}-y_i^{(k-2)}|\right.\right.$$

$$\left.\left.+L_2\int_{t_i}^{s}\max_i|y_i^{(k-1)}-y_i^{(k-2)}|d\tau\right\}ds\right|$$

$$\leq L_1\left|\int_{t_i}^{t}\left\{\max_i|y_i^{(k-1)}-y_i^{(k-2)}|\right.\right.$$

$$\left.\left.+L_2\int_{t_i}^{t}\max_i|y_i^{(k-1)}-y_i^{(k-2)}|d\tau\right\}ds\right|,$$

for $k=2$ and $1\leq i\leq n$, we obtain

$$|y_i^{(2)}(t)-y_i^{(1)}(t)|\leq M_1L_1\frac{(t-t_1)^2}{2\,!}[1+L_2(t-t_i)],\ (t_i\leq t\leq T),$$

$$|y_i^{(2)}(t)-y_i^{(1)}(t)|\leq M_1L_1\frac{(t_n-t)^2}{2\,!}[1+L_2(t_i-t)],\ (0\leq t\leq t_i).$$

Similarly, if $k=3$, then for $1\leq i\leq n$, we have

$$|y_i^{(3)}(t)-y_i^{(2)}(t)|\leq M_1L_1^2\frac{(t-t_1)^3}{3\,!}[1+L_2(t-t_1)],\ [1+L_2(t-t_i)],\ (t_i\leq t\leq T),$$

$$|y_i^{(3)}(t)-y_i^{(2)}(t)|\leq M_1L_1^2\frac{(t_n-t)^3}{3\,!}[1+L_2(t_n-t)]\,[1+L_2(t_i-t)],\ (0\leq t\leq t_i).$$

Again, using mathematical induction for $1\leq i\leq n$, we have

$$|y_i^{(k)}(t)-y_i^{(k-1)}(t)|\leq M_1L_1^{k-1}\frac{(t-t_1)^k}{k\,!}[1+L_2(t-t_1)]^{k-2}.\ [1+L_2(t-t_i)],$$

$$(t_i\leq t\leq T) \qquad (3)$$

$$|y_i^{(k)}(t)-y_i^{(k-1)}(t)|\leq M_1L_1^{k-1}\frac{(t_n-t)^k}{k\,!}[1+L_2(t_n-t)]^{k-2}.\ [1+L_2(t_i-t)],$$

$$(0\leq t\leq t_i) \qquad (4)$$

Therefore, the sequence of approximation $\{y_i^{(k)}(t)\}\,(0\leq t\leq T)$ converges uniformly, i.e.

$$\lim_{k\to\infty} y_i^{(k)}(t)=y_i(t).$$

Upon integrating the equations in (2) and taking the limit as k becomes very large (∞), we deduce that $y_i(t)$ $(1\leq i\leq n)$ represent a solution for problem (1).

The uniqueness of the solution of problem (1), under the conditions a), b) and c), and be proved by contradiction [3], [5].

ESTIMATING THE RATE OF CONVERGENCE 2.2. From systems (1), (2) we have

$$\begin{aligned} |y_i^{(k)}(t)-y_i(t)| &\leq L_1\left|\int_{t_i}^{t}\left\{\max_i|y_i^{(k-1)}-y_i|\right.\right. \\ &\quad \left.\left.+L_2\int_{t_i}^{s}\max_i|y_i^{(k-1)}-y_i|d\tau\right\}ds\right| \\ &\leq L_1\left|\int_{t_i}^{t}\left\{\max_i|y_i^{(k-1)}-y_i|\right.\right. \\ &\quad \left.\left.+L_2\int_{t_i}^{t}\max_i|y_i^{(k-1)}-y_i|d\tau\right\}ds\right|. \end{aligned} \tag{5}$$

For $k=1$ and $1\leq i\leq n$, the inequality (5) leads to

$$|y_i^{(1)}(t)-y_i(t)|\leq RL_1(t-t_i)[1+L_2(t-t_i)],\quad (t_i\leq t\leq T),$$
$$|y_i^{(1)}(t)-y_i(t)|\leq RL_1(t_i-t)[1+L_2(t_i-t)],\quad (0\leq t\leq t_i).$$

Using mathematical induction and the set of conditions a), b) and c), the rate of convergence for Picard's approximations can be estimated as follows:

$$|y_i^{(k)}(t)-y_i(t)|\leq R\frac{[L_1(t-t_1)]^k}{k\,!}[1+L_2(t-t_1)]^{k-1}.\ [1+L_2(t-t_i)],\quad (t_i\leq t\leq T), \tag{6}$$

$$|y_i^{(k)}(t)-y_i(t)|\leq R\frac{[L_1(t_n-t)]^k}{k\,!}[1+L_2(t_n-t)]^{k-1}.\ [1+L_2(t_i-t)],\quad (0\leq t\leq t_i). \tag{7}$$

Thus we achived the following theorem.

THEOREM 1. *If conditions a), b) and c) are satisfied, then problem* (1) *has a unique solution which is the limit of Picard's successive approximations* (2) *and the rate of convergence for any* $i(1\leq i\leq n)$ *is determined by the inequalities* (6), (7).

3. Seidel's method of succesaive approximations

For solving problem (1) numerically, we construct Seidel's successive approximations as follows:

$$\left.\begin{aligned} & y_i^{(0)}(t)\ \text{(the intial approximations)}=\alpha_i,\ y_i^{(k)}(t_i)=\alpha_i,\ (i=\overline{1,n}),\\ & \frac{dy_i^{(k)}(t)}{dt}=f_i[t,\ y_1^{(k)},\ y_2^{(k)},\ \cdots,\ y_{i-1}^{(k)},\ y_i^{(k-1)},\ \cdots,\ y_n^{(k-1)}\\ &\qquad ;\ \int_{t_i}^{t} g_i(t,\ s,\ y_1^{(k)},\ \cdots,\ y_{i-1}^{(k)},\ y_i^{(k-1)},\ \cdots,\ y_n^{(k-1)})ds],\\ &\qquad\qquad k=1,\ 2,\ \cdots \end{aligned}\right\} \quad (8)$$

THE SUFFICIENT CONDITIONS FOR CONVERGENCE 3.1. In addition to the set of conditions a), b) and c), we assume that the following condition is satisfied:

d) $L_1T(1+L_2T)<1$.

Using the condition c), then for $k=0,\ 1,\ 2,\ \cdots$, it can be proved that

$$|y_i^{(k)}(t)-\alpha_i|\leq R,\quad (i=\overline{1,n}).$$

It is obvious that:

$$|y_i^{(1)}(t)-y_i^{(0)}(t)|=|y_i^{(1)}(t)-\alpha_i|\leq M_1|t-t_i|,\ \ (i=\overline{1,n}).$$

For $t_1\leq t\leq T$, we have

$$\begin{aligned}|y_1^{(2)}(t)-y_1^{(1)}(t)| &\leq L_1\int_{t_1}^{t}\left\{\max_i|y_i^{(1)}-y_i^{(0)}|+L_2\int_{t_1}^{s}\max_i|y_i^{(1)}-y_i^{(0)}|d\tau\right\}ds\\ &\leq L_1\int_{t_1}^{t}\left\{\max_i|y_i^{(1)}-y_i^{(0)}|+L_2\int_{t_1}^{t}\max_i|y_i^{(1)}-y_i^{(0)}|d\tau\right\}ds\\ &\leq M_1L_1\frac{(t-t_1)^2}{2\,!}[1+L_2(t-t_1)],\end{aligned}$$

and for $0\leq t\leq t_1$ we get

$$|y_1^{(2)}(t)-y_1^{(1)}(t)|\leq M_1L_1\frac{(t_n-t)^2}{2\,!}.\ \ [1+L_2(t_1-t)].$$

In this way, it can be deduced that:

$$|y_i^{(2)}(t)-y_i^{(1)}(t)|\leq M_1L_1\frac{(t_1-t)^2}{2\,!}[1+L_2(t-t_i)],\ \ (t_i\leq t\leq T,\ \ i=\overline{1,n}),$$

$$|y_i^{(2)}(t)-y_i^{(1)}(t)|\leq M_1L_1\frac{(t_n-t)^2}{2!}[1+L_2(t_i-t)],\quad (0\leq t\leq t_i,\ i=\overline{1,n}).$$

Using mathematical induction and the condition d), we obtain

$$|y_i^{(k)}(t)-y_i^{(k-1)}(t)|\leq M_1L_1^{k-1}\frac{(t-t_1)^k}{k!}[1+L_2(t_i-t_1)]^{k-2}.\ [1+L_2(t-t_i)],$$
$$(t_i\leq t\leq T),$$

$$|y_i^{(k)}(t)-y_i^{(k-1)}(t)|\leq M_1L_1^{k-1}\frac{(t_n-t)^k}{k!}[1+L_2(t_n-t)]^{k-2}.\ [1+L_2(t_i-t)],$$
$$(0\leq t\leq t_i).$$

Therefore, the sequence of approximations $\{y_i^{(k)}(t)\}$ $(0\leq t\leq T)$ that difined by equations (8), converges uniformly. Integrating the equations in system (8) and then passing to the limit as $k\longrightarrow\infty$, we deduce that Seidel's successive approximations converge to the exact solution of problem (1).

ESTIMATING THE RATE OF CONVERGENCE 3.2. Using the set of conditions a), b), c) and d), we get

$$|y_1^{(1)}(t)-y_1(t)|\leq L_1\int_{t_1}^{t}\left\{\max_i|y_i^{(0)}-y_i|+L_2\int_{t_1}^{s}\max_i|y_i^{(0)}-y_i|d\tau\right\}ds$$
$$\leq RL_1(t-t_1)[1+L_2(t-t_1)],\quad (t_1\leq t\leq T),$$
$$|y_1^{(1)}(t)-y_1(t)|\leq RL_1(t_n-t)\ [1+L_2(t_1-t)],\quad (0\leq t\leq t_1).$$

Taking into account the condition d), we have

$$|y_i^{(1)}(t)-y_i(t)|\leq RL_1(t-t_1)\ [1+L_2(t-t_i)],\quad (t_i\leq t\leq T,\ i=\overline{1,n}),$$
$$|y_i^{(1)}(t)-y_i(t)|\leq RL_1(t_n-t)\ [1+L_2(t_i-t)],\quad (0\leq t\leq t_i,\ i=\overline{1,n}).$$

By mathematical induction, for $1\leq i\leq n$, we obtain

$$|y_i^{(k)}(t)-y_i(t)|\leq R\frac{[L_1(t-t_1)]^k}{k!}[1+L_2(t-t_1)]^{k-1}.\ [1+L_2(t-t_i)],$$
$$(t_i\leq t\leq t),\tag{9}$$

$$|y_i^{(k)}(t)-y_i(t)|\leq R\frac{[L_1(t_n-t)]^k}{k!}[1+L_2(t_n-t)]^{k-1}.\ [1+L_2(t_i-t)].$$
$$(0\leq t\leq t_i).\tag{10}$$

Hence the following theorem is proved.

THEOREM 2. *If the conditions a), b), c) and d) are satisfied, then problem*

(1) *has a unique solution which is obtained as the limit of Seidel's successive approximations. Furthermore, the rate of convergence is estimated by*(9), (10).

REMARK. In the case of $[L_1T(1+L_2T)]\geq 1$, is can be proved that the sequence of Seidel's approximations $\{y_i^{(k)}(t)\}$, $(0\leq t\leq T,\ 1\leq i\leq n,\ k=0,1,\cdots)$ diverges.

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