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A NOTE ON OPERATORS IN THE CLASSES $A_{m,n}$

By I. B. Jung and C. Y. Park

Let $\mathscr{H}$ be a separable, infinite dimensional, complex Hilbert space and let $\mathcal{L}(\mathscr{H})$ be the algebra of all bounded linear operators on $\mathscr{H}$. A *dual algera* is a subalgebra of $\mathcal{L}(\mathscr{H})$ that contains I and is closed in the weak* topology on $\mathcal{L}(\mathscr{H})$. Suppose m and n are cardinal numbers such that $1\leq m,\ n\leq\aleph_0$. A dual algebra $\mathscr{A}$ will be said to have property $(A_{m,n})$ if every $m\times n$ system of simultaneous equations of the form

$$(1)\qquad [x_i\otimes y_j]=[L_{ij}],\ \ 0\leq i<m,\ \ 0\leq j<n,$$

where $\{[L_{ij}]\}_{\substack{0\leq i<m\\ 0\leq j<n}}$ is an arbitrary $m\times n$ array from $Q_{\mathscr{A}}$, has a solution $\{x_i\}_{0\leq i<m,}$ $\{y_j\}_{0\leq j<n}$ consisting of a pair of sequences of vectors from $\mathscr{H}$. Furthermore, if m and n are positive integers and r is a fixed real number satisfying $r\geq 1$, a dual algebra $\mathscr{A}$ (with property$(A_{m,n})$) is said to have propery $(A_{m,n}(r))$ if for every $s>r$ and every $m\times n$ array $\{[L_{ij}]\}_{\substack{0\leq i<m\\ 0\leq j<n}}$ from $Q_{\mathscr{A}}$, there exist sequences $\{x_i\}_{0\leq i<m,}$ $\{y_j\}_{0\leq j<n}$ that satisfy (1) and also satisfy the following conditions:

$$(2)\qquad \|x_i\|^2\leq s\sum_{0\leq j<n}\|[L_{ij}]\|,\ \ 0\leq i<m,$$

and

$$(3)\qquad \|y_j\|^2\leq s\sum_{0\leq i<m}\|[L_{ij}]\|,\ \ 0\leq j<n.$$

Finally, a dual algebra $\mathscr{A}\subset\mathcal{L}(\mathscr{H})$ has property $(A_{m,\aleph_0}(r))$ (for some real number $r\geq 1$) if for every $s>r$ and every array $\{[L_{ij}]\}_{\substack{0\leq i<m\\ 0\leq j<\infty}}$ from $Q_{\mathscr{A}}$ with summable rows, there exist sequences $\{x_i\}_{0\leq i<m}$ and $\{y_j\}_{0\leq j<\infty}$ of vectors from $\mathscr{H}$ that satisfy (2) and (3) with the replacement of n by $\aleph_0$. Properties$(A_{\aleph_0,n}(r))$ and $(A_{\aleph_0,\aleph_0}(r))$ are defined similarly (cf. [5]).

We write $\boldsymbol{D}$ for the open unit disc in the complex plane $\boldsymbol{C}$ and $\boldsymbol{T}$ for the boundary of $\boldsymbol{D}$. A contraction $T\in\mathcal{L}(\mathscr{H})$ (i.e., $\|T\|\leq 1$) *is absolutely continuous* if in the canonical decomposition $T=T_1\oplus T_2$, where T_1 is a unitary operator and T_2 is a completely nonunitary contraction, T_1 is either absolutely continuous or acts on the space(0). We denote by $A=A(\mathscr{H})$ the class of all absolutely continuous contractions T in $\mathscr{L}(\mathscr{H})$ for which the Foias-Sz.-Nagy functional calculus $\Phi_T : H^\infty \to \mathscr{A}_T$ is an isometry (cf. [5]). Let φ_T be a bounded, linear, one-to-one map from Q_T into L^1/H^1_0 with $\varphi^*_T = \Phi_T$. Furthermore, if m and n are any cardinal numbers such that $1 \le m$, $n \le \aleph_0$, we denote by $A_{m,n} = A_{m,n}(\mathscr{H})$ the set of all T in $A(\mathscr{H})$ such that the singly generated dual algebra $\mathscr{A}_T$ has property $(A_{m,n})$. H. Bercovici(cf. [4]) and B. Chevreau(cf. [8]) proved independently that $A=A_1(1)$. I. Jung[12] showed that the classes $A_{m,n}$ are distinct one from another. C. Apostol, H. Bercovici, C. Foias and C. Pearcy[1], [2] studied geometric criteria for membership in the class $A_{\aleph_0} = A_{\aleph_0, \aleph_0}$. S. Brown, B. Chevreau, G. Exner and C. Pearcy[7], [9], [10], [11] obtained topological criteria and geometric criteria for membership in the class $A_{\aleph_0}$ or $A_{1,\aleph_0}$. In [13], I. Jung obtained some sufficient conditions for membership in the class $A_{1,\aleph_0}$ or $A_{\aleph_0}$. In this paper we obtain an equivalent condition for membership in the classes $A_{m,n}$. The notation and terminology employed herein agree with those in [3], [6] and [14].

Suppose $T \in A(\mathscr{H})$ and has minimal coisometric extension

$$B = S^* \oplus R$$

in $\mathscr{L}(\mathscr{K})$, where S is a unilateral shift of some multiplicity in $\mathscr{L}(\mathscr{S})$ if $\mathscr{S} \neq (0)$, $S=0$ if $\mathscr{S}=(0)$, R is a unitary operator if $\mathscr{U} \neq (0)$ and $R=0$ if $\mathscr{U}=(0)$ (cf. [10]). We shall use these notations for the following theorem.

THEOREM. *Suppose* $T \in A(\mathscr{H})$ *and* m *and* n *are any cardinal numbers such that* $1 \le m$, $n \le \aleph_0$. *Then* $T \in A_{m,n}$ *if and only if, for* $\{[L_{ij}]\}_{\substack{0 \le i < m \\ 0 \le j < n}} \subset Q_T$, *there exist sequences* $\{x^{(i)}_k\}^\infty_{k=1} \subset \mathscr{H}$, $\{w^{(j)}_k\}^\infty_{k=1} \subset \mathscr{S}$, *and* $\{b^{(j)}_k\}^\infty_{k=1} \subset \mathscr{U}$, *such that* $\{\|w^{(i)}_k + b^{(j)}_k\|\}$ *is bounded*, $\{x^{(i)}_k\}$ *Cauchy sequence and*

$$\|(\varphi_B^{-1} \cdot \varphi_T)([L_{ij}]_T) - [x^{(i)}_k \otimes (w^{(j)}_k + b^{(j)}_k)]_B\| \to 0.$$

PROOF. The idea of this proof comes from [10, Theorem 4.4]. Suppose $T \in A_{m,n}(\mathscr{H})$. It follows from the definition of property $(A_{m,n})$ that, for $\{[L_{ij}]\}_{\substack{0 \le i < m \\ 0 \le j < n}}$ in Q_T, there exist $x^{(i)}$, $y^{(j)} \in \mathscr{H}$, $0 \le i < m$, $0 \le j < n$, such that

$$[L_{ij}]_T=[x^{(i)}\otimes y^{(j)}]_T.$$

Set

$$x_k^{(i)}=x^{(i)},\ \ y_k^{(j)}=y^{(j)}=w^{(j)}+b^{(j)}\in\mathscr{S}\oplus\mathscr{W}$$

for any $k\in\boldsymbol{N}$. Then it is obvious that these are required sequences. Conversely, let us

$$x_k^{(j)}=P(w_k^{(j)}+b_k^{(j)}),\ \ k\in\boldsymbol{N},$$

where P is an orthogonal projection from $\mathscr{K}$ onto $\mathscr{U}$. Since $\{v_k^{(j)}\}$ is bounded, we may suppose w. l. o. g. that $\{a_k^{(j)}\}_{k=1}^{\infty}$ converges weakly to $v^{(j)}$. Moreover, since $\{x_k^{(i)}\}$ is a Cauchy sequence, we have $\{x_k^{(i)}\}$ converges strongly to $x^{(i)}$. Since $\{v_k^{(j)}\}$ is bounded, we have

$$\begin{aligned}\|[x^{(i)}\otimes v_k^{(j)}]-[x_k^{(i)}\otimes v_k^{(j)}]\|&=\|(x^{(i)}-x_k^{(i)})\otimes v_k^{(j)}\|\\&\le\|x^{(i)}-x_k^{(i)}\|\ \|v_k^{(j)}\|\to 0.\end{aligned}$$

Also we have

$$\begin{aligned}\|[L_{ij}]_T-[x_k^{(i)}\otimes v_k^{(j)}]_T\|&=\|\varphi_B^{-1}\cdot\varphi_T([L_{ij}]_T)-[x_k^{(i)}\otimes v_k^{(j)}]_B\|\\&=\|\varphi_B^{-1}\cdot\varphi_T([L_{ij}]_T)-[x_k^{(i)}\otimes(w_k^{(j)}+b_k^{(j)})]_B\|\to 0.\end{aligned}$$

Then

$$\begin{aligned}\|[L_{ij}]_T-[x^{(i)}\otimes v_k^{(j)}]_T\|&\le\|[L_{ij}]_T-[x_k^{(i)}\otimes v_k^{(j)}]_T\|\\&\quad+\|[x_k^{(i)}\otimes v_k^{(j)}]_T-[x^{(i)}\otimes v_k^{(j)}]_T\|\to 0.\end{aligned}$$

So

$$\|[L_{ij}]_T-[x^{(i)}\otimes v_k^{(j)}]_T\|\to 0.$$

We now compute to show that

$$[L_{ij}]_T=[x^{(i)}\otimes v^{(j)}]_T,$$

and thus complete the proof; for $h\in H^{\infty}(T)$, we have

$$\begin{aligned}\langle h(T),\ [L_{ij}]\rangle&=\lim_k\langle h(T),\ [x^{(i)}\otimes v_k^{(j)}]_T\rangle\\&=\lim_k(h(T)x^{(i)},\ v_k^{(j)})\\&=(h(T)x^{(i)},\ v^{(j)})\\&=\langle h(T),\ [x^{(i)}\otimes v^{(j)}]_T\rangle.\end{aligned}$$

Hence

$$[L_{ij}]_T = [x^{(i)} \otimes v^{(j)}]_T.$$

Therefore the proof is complete.

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Department of Mathematics
College of Natural Science
Kyungpook National University
Taegu 702-701, Korea