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ON SUBCLASSES OF P-VALENT FUNCTIONS STARLIKE IN THE UNIT DISC

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Abstract For a positive integer p, A_p will denote the class of functions $f(z)=z^p+\sum_{n=p+1}^{\infty} a_n z^n$ which are analytic in the unit disc $U=\{z:|z|<1\}$.

For $0\leq\alpha\leq 1, 0<\beta\leq 1, 0\leq\lambda<p$, let $S_p(\alpha,\beta,\lambda)$ denote the class of functions $f(z)\in A_p$ which satisfy the condition

$$\left|\frac{\frac{zf'(z)}{f(z)}-p}{\alpha\frac{zf'(z)}{f(z)}+p-\lambda(1+\alpha)}\right|<\beta \text{ for } z\in U$$

In this paper we obtain a representation theorem for the class $S_p(\alpha,\beta,\lambda)$ and also derive distortion theorem and sharp estimates for the coefficients of this class.

1. Introduction

For a positive integer p, A_p will denote the class of functions $f(z)=z^p+\sum_{n=p+1}^{\infty} a_n z^n$ which are analytic in $U=\{z:|z|<1\}$. Further, let

$$\left|\frac{\frac{zf'(z)}{f(z)}-p}{\alpha\frac{zf'(z)}{f(z)}+p-\lambda(1+\alpha)}\right|<\beta \tag{1.1}$$

with $0\leq\alpha\leq 1$, $0<\beta\leq 1$, $0\leq\lambda<p$ and $z\in U$. We shall then say that $f(z)\in S_p(\alpha,\beta,\lambda)$. It is the aim of this paper to obtain a representation theorem for the class $S_p(\alpha,\beta,\lambda)$ and also derive distortion theorem and sharp estimates for the coefficients of this class. $S_p(1,1,0)=S_p$ and $S_p(1,1,\lambda)=S_p(\lambda)$ are respectively the classes of p-valent starlike functions and p-valent starlike functions of order $\lambda, 0\leq\lambda<p$. Also $S_1(0,1,0)$, $S_1(1,\beta,0)$ and $S_1(\alpha,\beta,0)$ are respectively the classes of functions studied by R. Singh [4], K.S. Padmanabhan [3] and T.V. Lakshminarasimhan [2].

We state the following lemma that is needed in our investigation.

LEMMA 1 [2]. *$f(z)\in S_1(\alpha,\beta,0)=S(\alpha,\beta), 0\leq\alpha\leq 1$ and $0<\beta\leq 1$, if and only if*

$$f(z)=z\exp\left[-(1+\alpha)\int_0^z \frac{\phi(t)}{1+\alpha t\phi(t)}dt\right],$$

where $\phi(z)$ is analytic in U and $|\phi(z)|\leq\beta$ for $z\in U$.

2. Representation formulas for the class $S_p(\alpha,\beta,\lambda)$.

LEMMA 2. *Let $H(z)$ be analytic in U and satisfy the condition*

$$\left|\frac{p-H(z)}{[p-\lambda(1+\alpha)]+\alpha H(z)}\right|<\beta, z\in U,\ 0\leq\alpha\leq 1,\ 0<\beta\leq 1,\ 0\leq\lambda<p \tag{2.1}$$

with $H(0)=p$. Then we have

$$H(z)=\frac{p-[p-\lambda(1+\alpha)]z\phi(z)}{1+\alpha z\phi(z)} \tag{2.2}$$

where $\phi(z)$ is analytic in U and $|\phi(z)|\leq\beta$ for $z\in U$.

Conversely, any function $H(z)$ given by (2.2) above is analytic in U and satisfies (2.1).

PROOF. Let $H(z)=p+c_1z+c_2z^2+\cdots$ and let (2.1) hold. Then if

$$h(z)=\frac{p-H(z)}{[p-\lambda(1+\alpha)]+\alpha H(z)},$$

then $h(z)$ is analytic and satisfies $|h(z)|<\beta$, $z\in U$ and $h(0)=0$.

Hence by Schwarz's lemma, $h(z)=z\phi(z)$, where $|\phi(z)|\leq\beta$ for $z\in U$. Thus

$$H(z)=\frac{p-[p-\lambda(1+\alpha)]h(z)}{1+\alpha h(z)}=\frac{p-[p-\lambda(1+a)]z\phi(z)}{1+\alpha z\phi(z)}$$

$|\phi(z)|\leq\beta$ for $z\in U$, which is (2.2). On the other hand, if $H(z)$ is given by (2.2), then $H(z)$ is analytic for $z\in U$, and since $|z\phi(z)|\leq\beta|z|<\beta$ for $z\in U$, we have $\left|\frac{p-H(z)}{[p-\lambda(1+\alpha)]+\alpha H(z)}\right|=|z\phi(z)|<\beta$ for $z\in U$, and the proof of lemma 2 is complete.

LEMMA 3. *$f(z)\in S_p(\alpha,\beta,\lambda)$ if and only if*

$$f(z)=z^p\left[\frac{f_1(z)}{z}\right]^{(p-\lambda)} \tag{2.3}$$

for some $f_1(z)\in S_1(\alpha,\beta,0)=S(\alpha,\beta)$.

PROOF. Let $f(z)=z^p\Big[\frac{f_1(z)}{z}\Big]^{(p-\lambda)}$ for $f_1(z)=z+\sum_{n=2}^{\infty} c_n z^n \in S(\alpha,\beta)$, $z\in U$. By direct computation, we obtain

$$\frac{\frac{zf'(z)}{f(z)}-p}{\alpha\frac{zf'(z)}{f(z)}+p-\lambda(1+\alpha)}=\frac{\frac{zf'_1(z)}{f_1(z)}-1}{\alpha\frac{zf'_1(z)}{f_1(z)}+1}$$

and the result follows from (1.1).

We now proceed to prove a theorem which gives a representation for functions of the class $S_p(\alpha,\beta,\lambda)$.

THEOREM. 1. *$f(z)\in S_p(\alpha,\beta,\lambda)$, if and only if*

$$f(z)=z^p\exp\Big[-(p-\lambda)(1+\alpha)\int_0^z\frac{\phi(t)}{1+\alpha t\phi(t)}dt\Big] \tag{2.4}$$

where $\phi(z)$ is analytic and satisfies $|\phi(z)|\leq\beta$, $0<\beta\leq 1$, $z\in U$.

PROOF. Let $f(z)\in S_p(\alpha,\beta,\lambda)$, $0\leq\alpha\leq 1$, $0<\beta\leq 1$, $0\leq\lambda<p$.

Then lemma 2 gives, with $z\in U$,

$$\frac{zf'(z)}{f(z)}=\frac{p-[p-\lambda(1+\alpha)]z\phi(z)}{1+\alpha z\phi(z)} \tag{2.5}$$

where $\phi(z)$ is analytic in U and $|\phi(z|\leq\beta$ for $z\in U$. This gives

$$\frac{f'(z)}{f(z)}-\frac{p}{z}=\frac{-(p-\lambda)(1+\alpha)\phi(z)}{1+\alpha z\phi(z)} \tag{2.6}$$

which at once gives (2.4) on integration from 0 to z. On the other hand, if $f(z)$ is given by(2.4), it follows that (2.5) holds with $\phi(z)$ as in lemma 2, so that $f(z)\in S_p(\alpha,\beta,\lambda)$ and the proof is complete.

REMARK. There is another proof of Theorem 1 which is an immediate consequence of lemmas 1 and 3.

3. Distortion theorems.

LEMMA 4. *Let $f(z)\in S_p(\alpha,\beta,\lambda)$. Then, for $z\in U$,*

$$\frac{p-\beta[p-\lambda(1+\alpha)]|z|}{1+\alpha\beta|z|}\leq Re\Big\{\frac{zf'(z)}{f(z)}\Big\}\leq\frac{p+\beta[p-\lambda(1+\alpha)]|z|}{1-\alpha\beta|z|}$$

PROOF. If $f(z)\in S_p(\alpha,\beta,\lambda)$, then by lemma 2,

$$w=\frac{zf'(z)}{f(z)}=\frac{p-[p-\lambda(1+\alpha)]z}{1+\alpha z}$$

with $|z|<\beta|z|$, $z\in U$, so that

$$\text{Re } w=\text{Re}\frac{p-[p-\lambda(1+\alpha)]z}{1+\alpha z}\begin{cases}\geq\dfrac{p-\beta[p-\lambda(1+\alpha)]|z|}{1+\alpha\beta|z|}\\ \leq\dfrac{p+\beta[p-\lambda(1+\alpha)]|z|}{1-\alpha\beta|z|}\end{cases}$$

This proves lemma 4.

THEOREM 2. *If* $f(z)\in S_p(\alpha,\beta,\lambda)$ with $0<\alpha\leq 1$, $0<\beta\leq 1$, $0\leq\lambda<p$, we have

$$|f(z)|\begin{cases}\leq|z|^p(1-\alpha\beta|z|)^{-(p-\lambda)\left(\frac{1+\alpha}{\alpha}\right)}\\ \geq|z|^p(1+\alpha\beta|z|)^{-(p-\lambda)\left(\frac{1+\alpha}{\alpha}\right)}\end{cases}\tag{3.1}$$

while if $f(z)\in S_p(0,\beta,\lambda)$, $0<\beta\leq 1$, $0\leq\lambda<p$,

$$|f(z)|\begin{cases}\leq|z|^p\exp[(p-\lambda)\beta|z|]\\ \geq|z|^p\exp[-(p-\lambda)\beta|z|]\end{cases}\tag{3.2}$$

PROOF. Since $f(z)\in S_p(\alpha,\beta,\lambda)$, we have

$$\frac{zf'(z)}{f(z)}=\frac{p-[p-\lambda(1+\alpha)]z\phi(z)z}{1+\alpha z\phi(z)}\tag{3.3}$$

where $\phi(z)$ is analytic and satisfies $|\phi(z)|\leq\beta$ for $z\in U$. From (3.3), we obtain

$$\frac{f'(z)}{f(z)}-\frac{p}{z}=\frac{-(p-\lambda)(1+\alpha)\phi(z)}{1+\alpha z\phi(z)}\tag{3.4}$$

Using the fact that $|\phi(z)|\leq\beta$ for $z\in U$, we get from (3.4)

$$\begin{aligned}\log\left(\left|\frac{f(z)}{z^p}\right|\right)&=\text{Re}\left(\log\left(\frac{f(z)}{z^p}\right)\right)\\ &=\text{Re}\int_0^z\left[\frac{f'(s)}{f(s)}-\frac{p}{s}\right]ds\\ &=\text{Re}\int_0^z\left[\frac{-(p-\lambda)(1+\alpha)\phi(te^{i\theta})e^{i\theta}}{1+\alpha te^{i\theta}\phi(te^{i\theta})}\right]dt\\ &\leq\text{Re}\int_0^{|z|}\frac{(p-\lambda)(1+\alpha)|\phi(te^{i\theta})|}{|1+\alpha te^{i\theta}\phi(te^{i\theta})|}dt\end{aligned}$$

$$\leq \begin{cases} \int_0^{|z|} \frac{(p-\lambda)(1+\alpha)\beta}{1-\alpha\beta t} dt = (p-\lambda)\left(\frac{1+\alpha}{\alpha}\right)\log(1-\alpha\beta t)^{-1}, 0<\alpha\leq 1, \\ \int_0^{|z|} (p-\lambda)\beta dt = (p-\lambda)\beta|z|, \ \alpha=0, \end{cases}$$

so that, the first part of (3.1) and (3.2) respectively follow at once. On the other hand, lemma 4 gives

$$\mathrm{Re}\left\{\frac{zf'(z)}{f(z)}\right\} \geq \frac{p-\beta[p-\lambda(1+\alpha)]|z|}{1+\alpha\beta|z|}, \ 0\leq\alpha\leq 1,$$

so that for $|z|=r$,

$$r\mathrm{Re}\left\{\frac{\partial}{\partial r}\left(\log\frac{f(z)}{z^p}\right)\right\} = \mathrm{Re}\left\{\frac{zf'(z)}{f(z)} - p\right\}$$

$$\geq \frac{p-\beta[p-\lambda(1+\alpha)]|z|}{1+\alpha\beta|z|} - p$$

$$= \frac{-(p-\lambda)\beta(1+\alpha)|z|}{1+\alpha\beta|z|}$$

and hence

$$\log\left|\frac{f(z)}{z^p}\right| = \mathrm{Re}\left(\log\frac{f(z)}{z^p}\right)$$

$$\geq \int_0^r \frac{-(p-\lambda)\beta(1+\alpha)}{1+\alpha\beta r'} dr'$$

$$= \begin{cases} -(p-\lambda)\left(\frac{1+\alpha}{\alpha}\right)\log(1+\alpha\beta r), \ 0<\alpha\leq 1, \\ -(p-\lambda)\beta r, \ \alpha=0, \end{cases}$$

so that the second part of (3.1) and (3.2) respectively, follow at once since $|z|=r$. For equality we may take

$$f(z) = z^p(1-\alpha\beta z)^{-(p-\lambda)\left(\frac{1+\alpha}{\alpha}\right)}, \ 0<\alpha\leq 1, \ 0<\beta\leq 1, \ 0\leq\lambda<p,$$

and

$$f(z) = z^p \exp((p-\lambda)\beta z), \ \alpha=0, \ 0<\beta\leq 1, \ 0\leq\lambda<p.$$

4. Coefficient Estimates.

LEMMA 5. *If integers p and m are greater than zero, $0\leq\alpha\leq 1$, $0<\beta\leq 1$ and $0\leq\lambda<p$, then*

$$\prod_{j=0}^{m-1} \frac{\beta^2[(1+\alpha)(p-\lambda)+\alpha j]^2}{(j+1)^2} = \frac{1}{m^2}\left\{\beta^2(1+\alpha)^2(p-\lambda)^2 + \sum_{k=1}^{m-1}[\beta^2[(1+\alpha)(p-\lambda)+\alpha k]^2 - k^2] x \prod_{j=0}^{k-1}\frac{\beta^2[(1+\alpha)(p-\lambda)+\alpha j]^2}{(j+1)^2}\right\} \tag{4.1}$$

PROOF. We prove the lemma by induction on m. For $m=1$, (4.1) is easily verified directly.

Next suppose that (4.1) is true for $m=q-1$. We have

$$\frac{1}{q^2}\left\{\beta^2(1+\alpha)^2(p-\lambda)^2+\sum_{k=1}^{q-1}[\beta^2[(1+\alpha)(p-\lambda)+\alpha k]^2-k^2]\times\prod_{j=0}^{k-1}\frac{\beta^2[(1+\alpha)(p-\lambda)+aj]^2}{(j+1)^2}\right\}$$

$$=\frac{1}{q^2}\left\{\beta^2(1+\alpha)^2(p-\lambda)^2+\sum_{k=1}^{q-2}[\beta^2[(1+\alpha)(p-\lambda)+\alpha k]^2-k^2]\times\prod_{j=0}^{k-1}\frac{\beta^2[(1+\alpha)(p-\lambda)+\alpha j]^2}{(j+1)^2}+[\beta^2[(1+\alpha)(p-\lambda)+\alpha(q-1)]^2-(q-1)^2]\times\prod_{j=0}^{q-2}\frac{\beta^2[(1+\alpha)(p-\lambda)+\alpha j]^2}{(j+1)^2}\right\}$$

$$=\prod_{j=0}^{q-2}\frac{\beta^2[(1+\alpha)(p-\lambda)+\alpha j]^2}{(j+1)^2}\cdot\left\{\frac{\beta^2[(1+\alpha)(p-\lambda)+\alpha(q-1)]^2}{q^2}\right\}$$

$$=\prod_{j=0}^{q-1}\frac{\beta^2[(1+\alpha)(p-\lambda)+\alpha j]^2}{(j+1)^2}$$

Thus (4.1) holds for $m=q$ which proves lemma 5.

THEOREM 3. *If* $f(z)=z^p+\sum_{n=p+1}^{\infty}a_n z_n\in S_p(\alpha,\beta,\lambda)$, *then*

$$|a_n|\le\prod_{k=0}^{n-(p+1)}\frac{\beta[(1+\alpha)(p-\lambda)+\alpha k]}{k+1} \tag{4.2}$$

for $n\ge p+1$ *and these bounds are sharp for all admissible* α,β, *and* λ *for each* n.

PROOF. As $f\in S_p(\alpha,\beta,\lambda)$, from Theorem 1, we have

$$\{\alpha zf'(z)+[p-\lambda(1+\alpha)]f(z)\}\ z\phi(z)=pf(z)-zf'(z) \tag{4.3}$$

where $|\phi(z)|\le\beta$ for $z\in U$. Thus if

$$\psi(z)=z\phi(z)=\sum_{k=0}^{\infty}b_{k+1}z^{k+1}$$

we have $|\psi(z)|\le\beta|z|$ for $z\in U$ and (4.3) may be written as

$$\sum_{k=0}^{\infty}[\{(1+\alpha)(p-\lambda)+\alpha k\}a_{p+k}z^k]\sum_{k=0}^{\infty}b_{k+1}z^{k+1}$$
$$=\sum_{k=0}^{\infty}(-k)a_{p+k}z^k \tag{4.4}$$

where $a_p=1$.

Equating coefficients of z^m on both sides of (4.4), we obtain

$$\sum_{k=0}^{m-1}\{(1+\alpha)(p-\lambda)+\alpha k\}a_{p+k}b_{m-k}=\{-m\}a_{p+m};$$

which shows that a_{p+m} on right hand side depends only on

$$a_p, a_{p+1}, \cdots, a_{p+(m-1)}$$

of left-hand side. Hence we can write

$$\sum_{k=0}^{m-1}[\{(1+\alpha](p-\lambda)+ak\}a_{p+k}z^k]\Psi(z)$$

$$=\sum_{k=0}^{\infty}(-k)a_{p+k}z^k+\sum_{k=m+1}^{\infty}A_kz^k \tag{4.5}$$

for $m=1,2,3,\cdots$ and proper choice of $A_k(k\geq 0)$. Denoting the right member of (4.5) by $G(z)$ and the factor multiplying $\Psi(z)$ in the left member of (4.5) by $F(z)$, (4.5) assumes the form

$$G(z)=F(z)\Psi(z) \text{ for } z\in U.$$

Since $|\Psi(z)|\leq\beta$ for $z\in U$ this yields for $0<r<1$,

$$\frac{1}{2n}\int_0^{2n}|G(re^{i\theta})|^2d\theta\leq\frac{\beta^2}{2n}\int_0^{2n}|F(re^{i\theta})|^2d\theta,$$

whence, using the definitions of $G(z)$ and $F(z)$

$$\sum_{k=0}^{m}k^2|a_{p+k}|^2r^{2k}+\sum_{k=m+1}^{\infty}|A_k|^2r^{2k}$$

$$\leq\beta^2\left\{\sum_{k=0}^{m-1}[(1+\alpha)(p-\lambda)+\alpha k]^2|a_{p+k}|^2r^{2k}\right\}. \tag{4.6}$$

Setting $r\longrightarrow 1$ in (4.6), the inequality (4.6) may be written as

$$\sum_{k=0}^{m-1}\{\beta^2[(1+\alpha)(p-\lambda)+\alpha k]^2-k^2\}|a_{p+k}|^2\geq m^2|a_{p+m}|^2 \tag{4.7}$$

Replacing $p+m$ by n in (4.7), we are let to

$$|a_n|^2\leq\frac{1}{(n-p)^2}\sum_{k=0}^{n-(p+1)}\{\beta[^2(1+\alpha)(p-\lambda)+\alpha k|^2-k^2\}|a_{p+k}|^2, \tag{4.8}$$

where $n\geq p+1$.

For $n=p+1$, (4.8) reduces

$$|a_{p+1}|^2\leq\beta^2(1+\alpha)^2(p-\lambda)^2$$

or

$$|a_{p+1}|\leq\beta(1+\alpha)(p-\lambda) \tag{4.9}$$

which is equivalent to (4.2).

To establish (4.2) for $n>p+1$, we will apply induction argument.

Fix n, $n\geq p+2$, and suppose (4.2) holds for $k=1,2,3,\cdots,n-(p+1)$. Then

$$|a_n|^2\leq\frac{1}{(n-p)^2}\left\{\beta^2(1+\alpha)^2(p-\lambda)^2+\sum_{k=1}^{n-(p+1)}[\beta^2[(1+\alpha)(p-\lambda)\right.$$

$$\left.+\alpha k]^2-k^2]x\prod_{j=0}^{k-1}\frac{\beta^2[(1+\alpha)(p-\lambda)+\alpha j]^2}{(j+1)^2}\right\}. \tag{4.10}$$

Thus from (4.8), (4.10) and lemma 5 with $m=n-p$, we obtain

$$|a_n|^2 \leq \prod_{j=0}^{n-(p+1)} \frac{\beta^2[(1+\alpha)(p-\lambda)+\alpha j]^2}{(j+1)^2}$$

This completes the proof of Theorem 3. This proof is based on a technique found in Clunie [1].

Equality holds in (4.2) for $n \geq p+1$ for the function $f(z) \in A_p$ defined by

$$\frac{2f'(z)}{f(z)} = \frac{p-[p-\lambda(1+\alpha)]\beta z}{1+\alpha\beta z}$$

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