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HAUSDORFF DIMENSION OF SYMMETRIC CANTOR SETS

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1. Introduction

Many ways of estimating the size of thin sets have been proposed. The first of these to be extensively developed was established by Hausdorff. Hausdorff dimension has the overriding advantge from mathematician's point of view that Hausdorff measure is an outer measure. However, the Hausdorff dimension of even relatively simple sets can be hard to calculate. Hausdorff showed that the Hausdorff dimension of the famous middle-third set of Cantor is $\log 2/\log 3$. Since then tremendous amount has been discovered about Hausdorff dimension. In this paper we prove a very useful criterion to calculate the Hausdorff dimension of symmetric Cantor sets.

2. Preliminary and notations

(2.1) Hausdorff Dimension. Suppose that E is a subset of R. For positive α and ε, put

$$H_{\alpha}^{\varepsilon}(E)=\inf\sum_{n=1}^{\infty}[\delta(E_n)]^{\alpha}$$

where the infimum extends over all contable coverings of E by sets E_n with diameter, $\delta(E_n)=\sup\{|x-y|: x,y\in E_n\}$, less than ε. We define $\delta(\phi)=0$. As ε decreases, the infimum extends over smaller classes, and so $H_{\alpha}^{\varepsilon}(E)$ does not decrease. Thus H_{α}^{ε} has a limit $H_{\alpha}(E)$, finite or infinite:

$$H_{\alpha}^{\varepsilon}(E)\uparrow H_{\alpha}(E) \text{ as } \varepsilon\downarrow 0.$$

It is easy to see that H_{α} is an outer measure. The Hausdorff dimension of E is defined by considering the behavior of $H_{\alpha}(E)$, not as a function of E, but as a function of α.

It is well known that there exists a unique point α_0 such that $H_{\alpha}(E)=\infty$ for $\alpha<\alpha_0$ and $H_{\alpha}(E)=0$ for $\alpha>\alpha_0$. This value α_0 is called the ***Hausdorff dimension*** of E(denoted by dim E).

(2.2) Symmetric Cantor Sets. Let $\mathbf{c}=(c_k)_{k=0}^{\infty}$ be a fixed sequence of real numbers such that

$$0<2c_n<c_{n-1} \text{ for } n\geq 1$$

and put

$$r_n=c_{n-1}-c_n \text{ for } n\geq 1.$$

Let $F_n=\{\Sigma_{j=1}^{n}\varepsilon_j r_j: \varepsilon=0 \text{ or } 1 \text{ for all } j\}$. Then it is clear that $|s-t|>c_n$ for $s\neq t$ in F_n. In particular then, F_n has exactly 2^n points. Next, put $E_n=\cup_{t\in F_n}[t, t+c_n]$ which, by the above, is a disjoint union of 2^n closed intervals of length c_n each. Note that for $t\in F_n$, we have $t\in F_{n+1}$, $t+r_{n+1}\in F_{n+1}$, and

$$t<t+c_{n+1}<t+r_{n+1}<t+r_{n+1}+c_{n+1}=t+c_n. \tag{1}$$

This shows that $E_{n+1}\subset E_n$ for all $n\geq 1$. The set $E=E_c=\bigcap_{n=1}^{\infty}E_n$ will be called the *symmetric Cantor set* (on $[0, c_0]$) determined by $\mathbf{c}$. We easily see that

$$E=\{\sum_{i=1}^{\infty}\varepsilon_i r_i: \varepsilon_i=0 \text{ or } 1 \text{ for all } i\}.$$

(2.3) The Lebesgue Function [4, 5]. Let $E=E_c$ be the symmetric Cantor set determined by $\mathbf{c}=(c_k)_{k=0}^{\infty}$ as in (2.2). The *Lebesgue function* determined by $\mathbf{c}$, $L=L_c$, is defined as follows: For $\Sigma_{i=1}^{\infty}\varepsilon_i r_i\in E$, let

$$L(\sum_{i=1}^{\infty}\varepsilon_i r_i)=\sum_{i=1}^{\infty}\frac{\varepsilon_i}{2^i}.$$

This defines L on E. Now L is extended to all of $[0, c_0]$ by noting first that for each fixed $n\geq 0$ and $t=\Sigma_{i=1}^{n}\varepsilon_i r_i$ in F_n (where $F_0=\{0\}$), we have

$$\begin{aligned} L(t+c_{n+1}) &= L(t+\sum_{i=n+2}^{\infty}r_i)=\sum_{i=1}^{n}\frac{\varepsilon}{2^i}+\sum_{i=n+2}^{\infty}\frac{1}{2^i} \\ &= \sum_{i=1}^{n}\frac{\varepsilon}{2^i}+\frac{1}{2^{n+1}}=L(t)+\frac{1}{2^{n+1}} \\ &= L(t+r_{n+1}). \end{aligned}$$

[compare (2.2)-(1)]. We define L on $(t+c_{n+1}, t+r_{n+1})$ to have the constant value that it has at both of the endpoints, namely, $L(t)+\frac{1}{2^{n+1}}$. Note that L is monotone nondecreasing and continuous on $[0, c_0]$ and L is constant on each interval of $[0, c_0]\setminus E$.

(2.4) The Modulus of Continuity. Let $I\subset R$ be an interval and let $f: I\to C$

be a continuous function. Define

$$w_f(t)=\sup\{|f(x_2)-f(x_1)|:\ x_1,\ x_2\in I \text{ and } |x_2-x_1|\leq t\} \text{ for } t\geq 0.$$

The function w_f is called the *modulus of continuity* of f.

3. Results

(3.1) THEOREM. [FROSTMAN] *Let E be a compact subset of R. Suppose that there exists $\mu\in M^+(E)(\mu\neq 0)$ such that $w_\mu(t)=\sup_{0<x_2-x_1\leq t}\mu([x_1, x_2])=0(t^\alpha)(t\downarrow 0)$ for some $\alpha>0$. Then $H_\alpha(E)$ is positive(so* dim *$E\geq\alpha$).*

PROOF. Choose $0<C<\infty$ such that $w_\mu(t)\leq Ct^\alpha$ for $t>0$. Consider any open cover $\{(u_i,\ v_i)\}_{i=1}^\infty$ of E. Then we have

$$\sum_{i=1}^\infty (v_i-u_i)^\alpha \geq \frac{1}{C}\sum_{i=1}^\infty w_\mu(v_i-u_i)\geq \frac{1}{C}\sum_{i=1}^\infty \mu([u_i,\ v_i])$$

$$\geq \frac{\mu(E)}{C}=\frac{\|\mu\|}{C}>0.$$

Given any cover $\{A_i\}_{i=1}^\infty$ of E with $\delta(A_i)<\varepsilon$ for all i and any $\theta>0$, choose, for each i, $u_i<v_i$ with $A_i\subset(u_i,\ v_i)$ and $(v_i-u_i)^\alpha<[\delta(A_i)]^\alpha+\dfrac{\theta}{2^i}$ by continuity of the function t^α at $t=\delta(A_i)$. Then

$$\sum_{i=1}^\infty [\delta(A_i)]^\alpha+\theta\geq\sum_{i=1}^\infty (v_i-u_i)^\alpha\geq\frac{\|\mu\|}{C}>0$$

so

$$H_\alpha(E)+\theta\geq H_\alpha^\varepsilon(E)+\theta>\frac{\|\mu\|}{C}.$$

Thus

$$H_\alpha(E)\geq\frac{\|\mu\|}{C}>0.$$

(3.2) THEOREM. *Let E be the symmetric Cantor set determined by some $\mathbf{c}=(c_k)_{k=0}^\infty$ and let L be the corresponding Lebesgue function. Suppose that*

$$t^\alpha=0(w_L(t))\ (t\downarrow 0) \text{ for some } \alpha>0.$$

Then $H_\alpha(E)$ is finite(so dim *$E\leq\alpha$).*

PROOF. First we show that

$$\frac{1}{2^k}\leq w_L(c_k)\leq\frac{2}{2^k}. \tag{1}$$

Since $L(c_k)-L(0)=\frac{1}{2^k}$, the first inequality is clear. Let $0\leq x\leq c_0-c_k$ be given. Since 0 and $c_0-c_k=\Sigma_{i=1}^k r_i$ are both in $F_k=\{\Sigma_{i=1}^k \varepsilon_i r_i: \varepsilon_i\in\{0, 1\}$ for all $i\}$, we may define

$$t_1=\max\{t\in F_k: t\leq x\}$$

and

$$t_2=\min\{t\in F_k: x\leq t\}.$$

If $x\notin E_k$, then $L(x)=L(t_2)$ so

$$L(x+c_k)-L(x)\leq L(t_2+c_k)-L(t_2)=\frac{1}{2^k}.$$

If $x\in E_k$, then $t_1\leq x\leq t_1+c_k<t_2$ and $L(t_2)=L(t_1+c_k)$ so

$$L(x+c_k)-L(x)\leq L(t_2+c_k)-L(t_2)+L(t_1+c_k)-L(t_1)=\frac{2}{2^k}.$$

Thus the second inequality in (1) holds too. Choose $0<C<\infty$ such that $t^\alpha\leq C\omega_L(t)$ for $0<t<1$. Since $E\subset E_j$ and E_j is the union of 2^j intervals of length c_j, the definition of $H_\alpha(E)$ shows that if $c_j<\varepsilon$, then $H_\alpha^\varepsilon(E)\leq 2^j(c_j)^\alpha$. We can take $\varepsilon=2c_j$ to obtain

$$\begin{aligned}H_\alpha(E)&\leq\underline{\lim}_{j\to\infty}2^j(c_j)^\alpha\\&\leq C\,\underline{\lim}_{j\to\infty}2^j w_L(c_j)\leq 2C.\end{aligned}$$

where the last inequality is from (1).

(3.3) THEOREM. *Let E be a symmetric Cantor set and let L be the corresponding Lebesgue function. If*

$$\frac{\ln w_L(t)}{\ln t}\to\alpha\in R \text{ as } t\downarrow 0,$$

then the Hausdorff dimension of E, dim E, *is equal to* α.

PROOF. Given $\varepsilon>0$, there exists $0<t_\varepsilon<1$ such that

$$\begin{aligned}0<t<t_\varepsilon&\Rightarrow\alpha-\varepsilon<\frac{\ln w_L(t)}{\ln t}<\alpha+\varepsilon\\&\Rightarrow t^{\alpha+\varepsilon}<w_L(t)<t^{\alpha-\varepsilon}.\end{aligned}$$

Frostman's Theorem, applied to the $\mu\in M([0, 1])$ for which $\mu([a, b])=L(b)-L(a)$ whenever $0\leq a\leq b\leq 1$, now shows that dim $E\geq\alpha-\varepsilon$ if $\alpha-\varepsilon>0$. Therefore dim $E \geq \alpha$ even if $\alpha=0$. On the other hand, (3.2) shows us that

$$\dim E \leq \alpha+\varepsilon, \text{ so } \dim E \leq \alpha.$$

(3.4) THEOREM. *Let E be the symmetric Cantor set determined by $(c_k)_{k=0}^{\infty}$. If either*

$$\text{(i) } \lim_{k\to\infty} \frac{k \ln 2}{-\ln c_k}=\alpha \qquad \textit{or} \qquad \text{(ii) } \lim_{k\to\infty} \frac{k \ln 2}{-\ln r_k}=\alpha,$$

then $\dim E=\alpha$.

PROOF. Notice that $0 \leq \alpha\ 1$ since $c_k < 2^{-k} c_0$ for all $k \geq 1$. Choose k_0 with $c_{k_0} < 1$. For given t with $0 < t < c_{k_0}$, choose $k \geq k_0$ such that $c_{k+1} \leq t < c_k$. Then

$$-\ln c_{k+1} \geq -\ln t > -\ln c_k > 0. \tag{1}$$

As seen in the proof of Theorem 3.2

$$2^{-j} \leq w_L(c_j) \leq 2^{1-j}$$

and so, since $-\ln$ is decreasing,

$$(j-1)\ln 2 \leq -\ln w_L(c_j) \leq j \ln 2 \text{ for all } j \geq 1. \tag{2}$$

Since w_L is nondecreasing, we also have

$$\begin{aligned} 0 < -\ln w_L(c_k) &\leq -\ln w_L(t) \\ &\leq -\ln w_L(c_{k+1}) \text{ for our } t \text{ and } k. \end{aligned} \tag{3}$$

Therefore, (1)—(3) yield

$$\begin{aligned} \frac{(k-1)\ln 2}{-\ln c_{k+1}} &\leq \frac{(k-1)\ln 2}{-\ln t} \leq \frac{-\ln w_L(c_k)}{-\ln t} \leq \frac{-\ln w_L(t)}{-\ln t} \\ &\leq \frac{-\ln w_L(c_{k+1})}{-\ln t} \leq \frac{(k+1)\ln 2}{-\ln t} \leq \frac{(k+1)\ln 2}{-\ln c_k} \end{aligned}$$

and so

$$\frac{k-1}{k+1} \cdot \frac{(k+1)\ln 2}{-\ln c_{k+1}} \leq \frac{\ln w_L(t)}{\ln t} \leq \frac{k+1}{k} \cdot \frac{k \ln 2}{-\ln c_k}.$$

Thus, if $\lim_{k\to\infty} \frac{k \ln 2}{-\ln c_k}=\alpha$ then $\lim_{t\downarrow 0} \frac{\ln w_L(t)}{\ln t}=\alpha$ whence, by (3.3) $\dim E=\alpha$. Notice that $c_k < c_{k-1}-c_k=r_k$ and $r_{k+1} < \Sigma_{j=k+1}^{\infty} r_j = c_k$. Thus, if k satisfies $r_k < 1$, then

$$0 < -\ln r_k < -\ln c_k < -\ln r_{k+1}$$

so

$$\frac{k}{k+1} \frac{(k+1)\ln 2}{-\ln r_{k+1}} < \frac{k \ln 2}{-\ln c_k} < \frac{k \ln 2}{-\ln r_k}.$$

Thus (ii)implies (i) and so dim $E=\alpha$ if (ii) holds.

(3.5) **Corollary.** *Let E be a symmetric Cantor set with constant ratio,* $\frac{c_{k+1}}{c_k}=\xi\left(<\frac{1}{2}\right)$ *for all* $k\geq 0$. *Then* dim $E=\frac{\ln 2}{-\ln \xi}$.

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