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ON A CLASS OF MULTIVALENT FUNCTIONS WITH NEGATIVE COEFFICIENTS

By S.L. Shukla, A.M. Chaudhary and S. Owa*

Summary. Let $T_\lambda^\alpha(p, A, B)$ denote the class of functions

$$f(z)=z^p-\sum_{k=1}^{\infty}|a_{p+k}|z^{p+k}$$

which are regular and p valent in the unit disc $U=\{z: |z|<1\}$ and satisfying the condition

$$\left|\frac{e^{i\alpha}\left\{\frac{f'(z)}{z^{p-1}}-p\right\}}{(A-B)\lambda p\cos\alpha-Be^{i\alpha}\left\{\frac{f'(z)}{z^{p-1}}-p\right\}}\right|<1,\ z\in U,$$

where $0<\lambda\leq 1$, $-\frac{\pi}{2}<\alpha<\frac{\pi}{2}$, $-1\leq A<B\leq 1$, $0<B\leq 1$ and $p\in N=\{1, 2, 3, \cdots\}$.

In this paper, we obtain sharp results concerning coefficient estimates, distortion theorem and radius of convexity for the class $T_\lambda^\alpha(p, A, B)$. It is further shown that the class $T_\lambda^\alpha(p, A, B)$ is closed under "arithmetic mean" and "convex linear combinations". We also obtain class preserving integral operators of the form

$$F(z)=\frac{p+c}{z^c}\int_0^z t^{c-1}f(t)dt,\ c>-p,$$

for the class $T_\lambda^\alpha(p, A, B)$. Conversely when $F(z)\in T_\lambda^\alpha(p, A, B)$, radius of p valence of $f(z)$ has also been determined.

1. Introduction

Let $T_\lambda^\alpha(p, A, B)$ denote the class of functions

$$f(z)=z^p-\sum_{k=1}^{\infty}|a_{p+k}|z^{p+k} \tag{1.1}$$

which are regular and p valent in the unit disc $U=\{z: |z|<1\}$ and satisfying

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the condition

$$(1.2)\qquad \left|\frac{e^{i\alpha}\left\{\frac{f'(z)}{z^{p-1}}-p\right\}}{(A-B)\lambda p\cos\alpha-Be^{i\alpha}\left\{\frac{f'(z)}{z^{p-1}}-p\right\}}\right|<1,\quad z\in U.$$

It is easy to see that condition (1.2) is equivalent to

$$e^{i\alpha}\frac{f'(z)}{z^{p-1}}=(1-\lambda)pe^{i\alpha}+\lambda p\cos\alpha\left[\frac{1+A\omega(z)}{1+B\omega(z)}\right]+i\lambda p\sin\alpha$$

where $\omega(z)$ is regular in U and satisfying the conditions

$$\omega(0)=0,\quad |\omega(z)|<1 \text{ for } z\in U.$$

By giving specific values to A, B, λ, p and α, we obtain the following important classes studied by various authors in earlier works:

(i) Taking $\lambda=1$ and $\alpha=0$, the class $T_\lambda^\alpha(p, A, B)$ coincides with the class $p^*(p, A, B)$ introduced by Shukla and Dashrath [3].

(ii) Taking $p=1$, $\lambda=1$, $\alpha=0$, $B=\beta$ and $A=(2\alpha-1)\beta$ where $0<\beta\leq 1$, $0\leq\alpha<1$, the class $T_\lambda^\alpha(p, A, B)$ coincides with the class $p^*(\alpha, \beta)$ studied by Gupta and Jain [2].

In the present paper firstly we obtain the necessary and sufficient condition in terms of coefficient estimate. Then we obtain coefficient estimates, distortion theorems, class preserving integral operator and closure properties.

In the rest of the paper we always assume $-1\leq A<B\leq 1$, $0<\lambda\leq 1$ and $\alpha\in\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. Our results generalize many known results obtained so far in this direction.

2. Coefficient estimates

THEOREM 1. *A function* $f(z)=z^p-\sum_{k=1}^{\infty}|a_{p+k}|z^{p+k}$ is in $T_\lambda^\alpha(p, A, B)$ *if and only if*

$$(2.1)\qquad \sum_{k=1}^{\infty}(p+k)(1+B)|a_{p+k}|\leq\lambda(B-A)p\cos\alpha.$$

The result is sharp.

PROOF. Let $|z|=1$. Then

$$|e^{i\alpha}\{f'(z)-pz^{p-1}\}|-|(A-B)\lambda p\cos\alpha\, z^{p-1}-Be^{i\alpha}\{f'(z)-pz^{p-1}\}|$$
$$=|-e^{i\alpha}\sum_{k=1}^{\infty}(p+k)|a_{p+k}|z^{p+k-1}|-|(A-B)\lambda p\cos\alpha\, z^{p-1}$$

$$+Be^{i\alpha}\sum_{k=1}^{\infty}(p+k)|a_{p+k}|z^{p+k-1}|$$

$$\leq\sum_{k=1}^{\infty}(1+B)(p+k)|a_{p+k}|-\lambda(B-A)p\cos\alpha\leq 0,$$

Since the inequality (2.1) hold.

Hence, by maximum modulus theorem, we can see that $f(z)$ is in the class $T_{\lambda}^{\alpha}(p, A, B)$. To prove the converse, let

$$\left|\frac{e^{i\alpha}\left\{\frac{f'(z)}{z^{p-1}}-p\right\}}{\lambda(A-B)p\cos\alpha-Be^{i\alpha}\left\{\frac{f'(z)}{z^{p-1}}-1\right\}}\right|$$

$$\left|\frac{-e^{i\alpha}\sum_{k=1}^{\infty}(p+k)|a_{p+k}|z^{p+k-1}}{\lambda(A-B)p\cos\alpha\, z^{p-1}+Be^{i\alpha}\sum_{k=1}^{\infty}(p+k)|a_{p+k}|z^{p+k-1}}\right|<1.$$

Since $|Re(z)|\leq|z|$ for all z, we have

$$Re\left\{\frac{e^{i\alpha}\sum_{k=1}^{\infty}(p+k)|a_{p+k}|z^{p+k-1}}{\lambda(B-A)p\cos\alpha-Be^{i\alpha}\sum_{k=1}^{\infty}(p+k)|a_{p+k}|z^{p+k-1}}\right\}<1 \tag{2.2}$$

Choose values of z on the line $\theta=\frac{-\alpha}{(p+k-1)}$ in the complex plane, so that $f(z)$ is real. Upon clearing the denominator in (2.2) and letting $z^{p+k-1}e^{i\alpha}\to 1$ through real values, we obtain

$$\sum_{k=1}^{\infty}(p+k)|a_{p+k}|\leq\lambda(B-A)p\cos\alpha-B\sum_{k=1}^{\infty}(p+k)|a_{p+k}|.$$

This completes the proof of the theorem.

Sharpness follows if we take

$$f(z)=z^{p}-\frac{\lambda(B-A)p\cos\alpha\, e^{-i\alpha}}{(p+k)(1+B)}z^{p+k}.$$

THEOREM 2. *If $f(z)$ is in the class $T_{\lambda}^{\alpha}(p, A, B)$, then*

$$|a_{p+k}|\leq\frac{\lambda(B-A)p\cos\alpha}{(p+k)}. \tag{2.3}$$

The result is sharp.

PROOF. Since $f(z)$ belongs to $T_{\lambda}^{\alpha}(p, A, B)$, we have

$$e^{i\alpha}\frac{f'(z)}{z^{p-1}}=(1-\lambda)pe^{i\alpha}+\lambda p\cos\alpha\left[\frac{1+A\omega(z)}{1+B\omega(z)}\right]+i\lambda p\sin\alpha$$

where $\omega(z)=\sum_{j=1}^{\infty}t_j z^j$ is regular in U, satisfies $\omega(0)=0$ and $|\omega(z)|<1$ for $z\in U$. Hence

$$e^{i\alpha}\left\{\frac{f'(z)}{z^{p-1}}-p\right\}=\left[-(B-A)\lambda p\cos\alpha-Be^{i\alpha}\left\{\frac{f'(z)}{z^{p-1}}-p\right\}\right]\omega(z)$$

or

$$(2.4)\quad -e^{i\alpha}\sum_{j=1}^{\infty}|a_{p+j}|(p+j)z^j=[-(B-A)\lambda p\cos\alpha+Be^{i\alpha}\sum_{j=1}^{\infty}|a_{p+j}|(p+j)z^j]\times[\sum_{j=1}^{\infty}t_j z^j].$$

Equating corresponding coefficients on both sides of above equation (2.4), we find that the coefficient a_{p+k} on the left hand side of (2.4) depends only on $a_{p+1}, a_{p+2}, \cdots, a_{p+k-1}$ on the right hand side of (2.4). Hence, for $k\geq 1$, it follows from (2.4) that

$$-e^{i\alpha}\sum_{j=1}^{k}|a_{p+j}|(p+j)z^j-e^{i\alpha}\sum_{j=k+1}^{\infty}|c_j|z^j=[-(B-A)\lambda p\cos\alpha+Be^{i\alpha}\sum_{j=1}^{k-1}|a_{p+j}|(p+j)z^j]\,\omega(z)$$

or

$$\sum_{j=1}^{k}|a_{p+j}|^2(p+j)^2|z|^{2j}+\sum_{j=k+1}^{\infty}|c_j|^2|z|^{2j}\leq\lambda^2(B-A)^2p^2\cos^2\alpha+B^2\sum_{j=1}^{k-1}|a_{p+j}|^2(p+j)^2|z|^{2j}$$

Letting $|z|=r\to 1$

$$\sum_{j=1}^{k}|a_{p+j}|^2(p+j)^2\leq\lambda^2(B-A)^2p^2\cos^2\alpha+B^2\sum_{j=1}^{k-1}|a_{p+j}|^2(p+j)^2$$

or

$$|a_{p+k}|^2(p+k)^2\leq\lambda^2(B-A)^2p^2\cos^2\alpha-(1-B^2)\sum_{j=1}^{k-1}|a_{p+j}|^2(p+j)^2$$

or

$$|a_{p+k}|^2\leq\frac{\lambda^2(B-A)^2p^2\cos^2\alpha}{(p+k)^2}$$

or

$$|a_{p+k}|\leq\frac{\lambda(B-A)p\cos\alpha}{(p+k)}.$$

In order to establish the sharpness, we consider the function f given by

$$e^{i\alpha}\frac{f'(z)}{z^{p-1}}=(1-\lambda)pe^{i\alpha}+\lambda p\cos\alpha\Big[\frac{1+Az^{p+k-1}}{1+Bz^{p+k-1}}+i\lambda p\sin\alpha.$$

Clearly $f(z)\in T_\lambda^\alpha(p,\ A,\ B)$.

It is easy to compute that the function $f(z)$ has the expansion

$$f(z)=z^p-\frac{\lambda(B-A)p\cos\alpha\, e^{-i\alpha}}{(p+k)}z^{p+k}.$$

THEOREM 3. *If $f(z)$ belongs to $T_\lambda^\alpha(p,\ A,\ B)$, then*

$$(1-B^2)\sum_{j=1}^{\infty}(p+j)^2|a_{p+j}|^2\le\lambda^2(B-A)^2p^2\cos^2\alpha. \tag{2.5}$$

PROOF. Since $f(z)$ belongs to $T_\lambda^\alpha(p,\ A,\ B)$, we have

$$e^{i\alpha}\frac{f'(z)}{z^{p-1}}=(1-\lambda)pe^{i\alpha}+\lambda p\cos\alpha\Big[\frac{1+A\omega(z)}{1+B\omega(z)}\Big]+i\lambda p\sin\alpha,$$

where $\omega(z)=\sum_{j=1}^{\infty}t_jz^j$ is regular in U, satisfies $\omega(0)=0$ and $|\omega(z)|<1$ for $z\in U$. Hence

$$e^{i\alpha}\Big\{\frac{f'(z)}{z^{p-1}}-p\Big\}=\Big[-(B-A)\lambda p\cos\alpha-Be^{i\alpha}\Big\{\frac{f'(z)}{z^{p-1}}-p\Big\}\Big]\omega(z)$$

or

$$-e^{i\alpha}\sum_{j=1}^{\infty}|a_{p+j}|(p+j)z^j=[-(B-A)\lambda p\cos\alpha+Be^{i\alpha}\sum_{j=1}^{\infty}|a_{p+j}|(p+j)z^j]\,[\sum_{j=1}^{\infty}t_jz^j]$$

By using parseval's identity, we get,

$$\sum_{j=1}^{\infty}|a_{p+j}|^2(p+j)^2r^{2j}\le\lambda^2(B-A)^2p^2\cos^2\alpha+B^2\sum_{j=1}^{\infty}|a_{p+j}|^2(p+j)^2r^{2j}$$

Letting $r\to1$, we get,

$$(1-B^2)\sum_{j=1}^{\infty}(p+j)^2|a_{p+j}|^2\le\lambda^2(B-A)^2p^2\cos^2\alpha.$$

3. Distortion Theorem

THEOREM 4. *If $f(z)\in T_\lambda^\alpha(p,\ A,\ B)$, then*

$$r^p-\frac{\lambda(B-A)p\cos\alpha}{(1+p)(1+B)}r^{p+1}\le|f(z)|\le r^p+\frac{\lambda(B-A)p\cos\alpha}{(1+p)(1+B)}r^{p+1} \tag{3.1}$$

and

(3.2) $$pr^{p-1}-\frac{\lambda(B-A)p\cos\alpha}{(1+B)}r^{p}\leq|f'(z)|\leq pr^{p-1}+\frac{\lambda(B-A)p\cos\alpha}{(1+B)}r^{p}.$$

The estimates are sharp.

PROOF. From theorem 1, we have

$$\sum_{k=1}^{\infty}(p+1)(1+B)|a_{p+k}|\leq\sum_{k=1}^{\infty}(p+k)(1+B)|a_{p+k}|\leq\lambda(B-A)p\cos\alpha$$

This implies that

$$\sum_{k=1}^{\infty}|a_{p+k}|\leq\frac{\lambda(B-A)p\cos\alpha}{(1+p)(1+B)}$$

Hence

$$|f(z)|\leq|z|^{p}+\sum_{k=1}^{\infty}|a_{p+k}|\,|z|^{p+k}\leq|z|^{p}+|z|^{p+1}\sum_{k=1}^{\infty}|a_{p+k}|$$
$$\leq r^{p}+\frac{\lambda(B-A)p\cos\alpha}{(1+p)(1+B)}r^{p+1}.$$

Similarly

$$|f(z)|\geq r^{p}-\frac{\lambda(B-A)p\cos\alpha}{(1+p)(1+B)}r^{p+1}.$$

Thus (3.1) follows.

Also

$$|f'(z)|\leq p|z|^{p-1}+\sum_{k=1}^{\infty}(p+k)|\,a_{p+k}|\,|z|^{p+k-1}$$
$$\leq pr^{p-1}+r^{p}\sum_{k=1}^{\infty}(p+k)|a_{p+k}|$$
$$\leq pr^{p-1}+\frac{\lambda(B-A)p\cos\alpha}{(1+B)}r^{p}.$$

Similarly

$$|f'(z)|\geq pr^{p-1}-\frac{\lambda(B-A)p\cos\alpha}{(1+B)}r^{p}.$$

This completes the proof of the theorem.

The bounds are sharp since the equalities are attained for the function

$$f(z)=z^{p}-\frac{\lambda(B-A)p\cos\alpha\, e^{-i\alpha}}{(p+1)(1+B)}z^{p+1},\quad (z=\pm r).$$

4. Integral Operators

THEOREM 5. *Let c be a real number such that $c>-p$. If $f(z)\in T_{\lambda}^{\alpha}(p, A, B)$, then the function $F(z)$ defined by*

(4.1) $$F(z)=\frac{p+c}{z^c}\int_0^z t^{c-1}f(t)dt$$

also belongs to $T_\lambda^\alpha(p, A, B)$.

PROOF. Let $f(z)=z^p-\sum_{k=1}^{\infty}|a_{p+k}|z^{p+k}$. Then from the representation of F, it follows that

$$F(z)=z^p-\sum_{k=1}^{\infty}|b_{p+k}|z^{p+k},$$

where $$|b_{p+k}|=\frac{(p+c)}{(p+c+k)}|a_{p+k}|.$$

Therefore

$$\begin{aligned}&\sum_{k=1}^{\infty}(p+k)(1+B)|b_{p+k}|\\ &=\sum_{k=1}^{\infty}(p+k)(1+B)\frac{(p+c)}{(p+c+k)}|a_{p+k}|\\ &\leq\sum_{k=1}^{\infty}(p+k)(1+B)|a_{p+k}|\\ &\leq\lambda(B-A)p\cos\alpha,\ \text{since } f(z)\in T_\lambda^\alpha(p, A, B).\end{aligned}$$

Hence, by theorem 1, $F(z)\in T_\lambda^\alpha(p, A, B)$.

THEOREM 6. *Let c be a real number, $c>-p$. If $F(z)\in T_\lambda^\alpha(p, A, B)$, then the function $f(z)$ defined in* (4.1) *is p valent for $|z|<R_p^*$, where*

$$R_p^*=\inf_{k\geq 1}\left[\left(\frac{p+c}{p+c+k}\right)\frac{(1+B)}{\lambda(B-A)\cos\alpha}\right]^{\frac{1}{k}}.$$

The result is sharp.

PROOF. Let $F(z)=z^p-\sum_{k=1}^{\infty}|a_{p+k}|z^{p+k}$. It follows then from (4.1) that.

$$\begin{aligned}f(z)&=\frac{z^{1-c}}{(p+c)}\frac{d}{dz}\{z^cF(z)\}\\ &=z^p-\sum_{k=1}^{\infty}\frac{(p+c+k)}{(p+c)}|a_{p+k}|z^{p+k}.\end{aligned}$$

To prove the result it suffices to show that

$$\left|\frac{f'(z)}{z^{p-1}}-p\right|\leq p \text{ for } |z|<R_p^*.$$

Now

$$\left|\frac{f'(z)}{z^{p-1}}-p\right|=\left|-\sum_{k=1}^{\infty}(p+k)\frac{(p+c+k)}{(p+c)}|a_{p+k}|z^k\right|$$

$$\leq\sum_{k=1}^{\infty}(p+k)\frac{(p+c+k)}{(p+c)}|a_{p+k}||z|^k$$

Thus

$$\left|\frac{f'(z)}{z^{p-1}}-p\right|\leq p \text{ if}$$

$$\sum_{k=1}^{\infty}(p+k)\frac{(p+c+k)}{(p+c)}|a_{p+k}||z|^k\leq p. \tag{4.2}$$

But theorem 1 confirms that

$$\sum_{k=1}^{\infty}\frac{(p+k)(1+B)|a_{p+k}|}{\lambda(B-A)\cos\alpha}\leq p.$$

Thus (4.2) will be satisfied if

$$(p+k)\frac{(p+c+k)}{(p+c)}|a_{p+k}||z|^k\leq\frac{(p+k)(1+B)|a_{p+k}|}{\lambda(B-A)\cos\alpha}$$

for each $k=1, 2, 3, \cdots$

or if

$$|z|\leq\left[\frac{(p+c)}{(p+c+k)}\ \frac{(1+B)}{\lambda(B-A)\cos\alpha}\right]^{\frac{1}{k}}. \tag{4.3}$$

The required result follows now from (4.3).

Sharpness follows if we take

$$F(z)=z^p-\frac{\lambda(B-A)p\cos\alpha\, e^{-ia}}{(p+k)(1+B)}z^{p+k},\ \text{for each } k=1, 2, 3, \cdots$$

5. Radius of Convexity

THEOREM 7. *If $f(z)\in T_{\lambda}^{\alpha}(p, A, B)$, then $f(z)$ is p valently convex in the disc*

$$|z|<R_p^{**}=\inf_k\left[\frac{(1+B)p}{\lambda(B-A)\cos\alpha(p+k)}\right]^{\frac{1}{k}},\ k=1, 2, 3, \cdots$$

The result is sharp.

PROOF. To prove the theorem, it is sufficient to show that

$$\left|\left(1+\frac{zf''(z)}{f'(z)}\right)-p\right|\leq p \text{ for } |z|<R_p^{**}.$$

We have

$$\left|1+\frac{zf''(z)}{f'(z)}-p\right|=\left|\frac{-\sum_{k=1}^{\infty}k(p+k)|a_{p+k}|z^{p+k-1}}{\{p-\sum_{k=1}^{\infty}(p+k)|a_{p+k}|z^{k}\}z^{p-1}}\right|$$

$$\leq\frac{\sum_{k=1}^{\infty}k(p+k)|\dot{a}_{p+k}||z|^{k}}{p-\sum_{k=1}^{\infty}(p+k)|a_{p+k}||z|^{k}}.$$

Thus

$$\left|\left\{1+\frac{zf''(z)}{f'(z)}\right\}-p\right|\leq p$$

if

$$\frac{\sum_{k=1}^{\infty}k(p+k)|a_{p+k}||z|^{k}}{p-\sum_{k=1}^{\infty}(p+k)|a_{p+k}||z|^{k}}\leq p$$

or

$$\sum_{k=1}^{\infty}\left(\frac{p+k}{p}\right)^{2}|a_{p+k}||z|^{k}\leq 1,$$

But from theorem 1, we obtain

$$\sum_{k=1}^{\infty}\frac{(p+k)(1+B)|a_{p+k}|}{\lambda(B-A)p\cos\alpha}\leq 1,$$

Hence $f(z)$ is p valently convex if

$$\left(\frac{p+k}{p}\right)^{2}|a_{p+k}||z|^{k}\leq\frac{(p+k)(1+B)|a_{p+k}|}{\lambda(B-A)p\cos\alpha}$$

or

$$|z|\leq\left[\frac{(1+B)p}{\lambda(B-A)\cos\alpha(p+k)}\right]^{\frac{1}{k}}\quad\text{for each } k=1,\ 2,\ 3,\ \cdots$$

This completes the proof of the theorem. The result is sharp for the function

$$f(z)=z^{p}-\frac{\lambda(B-A)p\cos\alpha\, e^{-i\alpha}}{(1+B)(p+k)}z^{p+k}.$$

6. Closure Properties

THEOREM 8. *If*

$$f(z)=z^{p}-\sum_{k=1}^{\infty}|a_{p+k}|z^{p+k}$$

and

$$g(z)=z^{p}-\sum_{k=1}^{\infty}|b_{p+k}|z^{p+k}$$

are in $T_{\lambda}^{\alpha}(p, A, B)$, *then*

$$h(z)=z^{p}-\frac{1}{2}\sum_{k=1}^{\infty}|a_{p+k}+b_{p+k}|z^{p+k}$$

also in $T_{\lambda}^{\alpha}(p, A, B)$.

PROOF. Since $f(z)$ and $g(z)$ are in $T_{\lambda}^{\alpha}(p, A, B)$ therefore we have

$$\sum_{k=1}^{\infty}(p+k)(1+B)|a_{p+k}|\leq\lambda(B-A)p\cos\alpha \tag{6.1}$$

and

$$\sum_{k=1}^{\infty}(p+k)(1+B)|b_{p+k}|\leq\lambda(B-A)p\cos\alpha. \tag{6.2}$$

From (6.1) and (6.2), we obtain

$$\frac{1}{2}\sum_{k=1}^{\infty}(p+k)(1+B)|a_{p+k}+b_{p+k}|\leq\lambda(B-A)p\cos\alpha$$

This completes the proof of the theorem.

THEOREM 9. *Let* $f_{p}(z)=z^{p}$,

$$f_{p+k}(z)=z^{p}-\frac{\lambda(B-A)p\cos\alpha}{(1+B)(p+k)}z^{p+k},\quad k=1, 2, 3, \cdots$$

Then $f(z)\in T_{\lambda}^{\alpha}(p, A, B)$ *if and only if it can be expressed in the form*

$$f(z)=\sum_{k=0}^{\infty}\lambda_{p+k}f_{p+k}(z)$$

where $\lambda_{p+k}\geq 0$ *and* $\sum_{k=0}^{\infty}\lambda_{p+k}=1$.

PROOF. Suppose

$$\begin{aligned}f(z)&=\sum_{k=0}^{\infty}\lambda_{p+k}f_{p+k}(z)\\&=z^{p}-\sum_{k=1}^{\infty}\frac{\lambda(B-A)p\cos\alpha}{(1+B)(p+k)}\lambda_{p+k}z^{p+k}.\end{aligned}$$

Then

$$\begin{aligned}&\sum_{k=1}^{\infty}\frac{(p+k)(1+B)}{\lambda(B-A)p\cos\alpha}\,\frac{\lambda(B-A)p\cos\alpha}{(1+B)(p+k)}\lambda_{p+k}\\&=\sum_{k=1}^{\infty}\lambda_{p+k}=1-\lambda_{p}\leq 1.\end{aligned}$$

Hence by theorem 1, $f(z)\in T_{\lambda}^{\alpha}(p, A, B)$.

Conversely, suppose that $f(z)\in T_{\lambda}^{\alpha}(p, A, B)$.

Since

$$|a_{p+k}|\leq\frac{\lambda(B-A)p\cos\alpha}{(p+k)(1+B)}, \text{ for each } k=1, 2, \cdots$$

we may set

$$\lambda_{p+k}=\frac{(1+B)(p+k)}{\lambda(B-A)p\cos\alpha}|a_{p+k}|, \text{ for each } k=1, 2, 3, \cdots$$

and

$$\lambda_p=1-\sum_{k=1}^{\infty}\lambda_{p+k}.$$

Then

$$f(z)=\sum_{k=0}^{\infty}\lambda_{p+k}f_{p+k}(z).$$

This completes the proof of the theorem.

References

[1] R.M. Goel and N.S. Sohi, "*Multivalent functions with negative coefficients*", Indian J. pure appl. Math. 12(7), (1981), 844—853.

[2] V.P. Gupta and P.K. Jain, "*Certain classes of univalent functions with negative coefficients II*", Bull. Austral. Math. Soc. 15(1976), 467—473.

[3] S.L. Shukla and Dashrath, "*On certain classes of multivalent functions with negative coefficients*", Soochow J. Math. and Natural Science 8(1982) 179—188.

Department of Mathematics
Janta College Bakewar
Etawah 206124, INDIA.

*Department of Mathematics
Kinki University
Higashi-Osaka, 577, Japan.