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THE QUASI-REVERSIBILITY METHOD APPLICATION TO PARABOLIC OPERATORS WITH AN INFINITE NUMBER OF VARIABLES

By S. A. El-Zahaby

Abstract: In the present paper, using the method of Quasi-Reversibility of R. Lattes and J. L. Lions [7], we have applied this method for a system governed by parabolic operator with an infinite number of variables. (The control here is an initial condition).

Introduction

I. M. Gali et al presented in [5] a set of inequalities defining an optimal control of a system governed by a selfadjoint elliptic operator with an infinite number of variables

$$(Au)(x) = -\sum_{k=1}^{\infty}(D_k^2 u)(x) + q(x)u(x),\ q(x) \geq \nu,\ 1 \geq \nu > 0 \tag{1}$$

where

$$(D_k u)(x) = \frac{1}{\sqrt{p_k(x_k)}} \frac{\partial}{\partial x_k}(\sqrt{p_k(x_k)}\,u(x))$$

and $q(x)$ is a real valued function from space of functions of infinitely many variables $L_2(R^\infty,\ dg(x))$. constructed by the measure $dg(x) = p(x_1)dx_1 \otimes p(x_2)dx^2$ $\otimes\cdots$defined on the σ hull of cylindrical sets in R^∞ generated by finite dimensional Borel sets; $g(R^\infty) = 1$.

$$\|u\|_{L_2(R^\infty,\ dg(x))} = \left(\int_{R^\infty} |u|^2 dg(x)\right)^{1/2} < \infty$$

More details for such construction see [2] and [3].

The system here is ruled by the operator

$$\frac{\partial}{\partial t} + A(t)$$

where $A(t)$ has the form (1).

The main method consists of solving the problem

$$\left(\frac{\partial}{\partial t}+A\right)u=0$$
$$u=0 \text{ on } \Sigma$$

$u(x,T)$=the solution at the final time$=X(x)$.

In general, there exists no ξ such that $u_\xi=X$ this is connected with the irreversibility of the problem. But on the other hand however small η we can always find ξ such that u_ξ approximates to X within η, this is connected with the backwards uniqueness. The problem is then to find one ξ yielding such an approximation.

Some Concepts and Results

Let us consider the elliptic operator

$$A(t)\phi(x, t)=-\sum_{k=1}^{\infty}D_k^2\phi(x, t)+q(x, t)\phi(x, t)$$

where

$$(D_k\phi)(x, t)=p_k^{-1/2}\frac{\partial}{\partial x_k}(p_k^{1/2}\phi) \tag{2}$$

$q(x, t)$ is a real valued function in x which is bounded and measurable on R such that $q(x, t)\geq\lambda>0$, λ is a constant.

We have the following chain [9]

$$L_2(0, T; W_0^1(R^\infty))\subseteq L_2(0, T; L_2(R^\infty))\subseteq L_2(0, T; W_0^{-1}(R^\infty))$$

The space $L_2(0, T; L^2(R^\infty))=L_2(Q)$, $Q=R^\infty x]0, T[$ $\Sigma=\Gamma x[0, T]$, Σ is the lateral boundary of Q.

For each t we may write a continuous bilinear form

$$\pi(t; u, v)=(A(t)u, v),\ u, v\in W_0'(R^\infty) \tag{3}$$

where $A(t)$ is a bounded self-adjoint elliptic operator with an infinite number of variables maps $W_0^1(R^\infty)$ onto $W_0^{-1}(R^\infty)$.

This bilinear form is coercive [4-6] in $W_0^1(R^\infty)$ that means

$$\pi(t; u, u)\geq\nu\|u\|^2_{W_0'(R^\infty)}\quad \nu>0 \tag{4}$$

For all u, $v\in W_0^1(R^\infty)$ the function $t\to\pi(u, v)$ is measurable and continuous on $]0, T[$ and

$$\pi(t; u, v)=\pi(t; v, u) \tag{5}$$

From the above consideration and from [4], [5], [6], and [7], we can formulate the following:

THEOREM 1. *Under the hypotheses* (4) *and* (5) *if* f *and* ξ *are given in* $L_2(0, T; W_0^{-1}(R^\infty))$ *and* $L_2(R^\infty)$, *respectively then there exists a unique element* u *that satisfies*

$$\frac{du}{dt}+A(t)u=f \qquad u\in L_2(0, T; W_0^1(R^\infty)),$$
$$u(0, x)=\xi \qquad u'\in L_2(0, T; W_0^{-1}(R^\infty))$$
$$u=0 \qquad \text{on } \Sigma$$

Formulation of the Problem

Let X be a given function in $L_2(R^\infty)$ and $T>0$ be given. To each ξ corresponding the solution $u(x, T; \xi)$ of

$$\frac{\partial u}{\partial t}+A(t)u=0$$
$$u(0, x)=\xi$$
$$u=0 \qquad \text{on } \Sigma$$
$$u\in L_2(Q), \qquad u'\in L_2(Q)$$

Now, we set

$$J(\xi)=\int_{R^\infty}|u(x, T)-X|^2dx \tag{6}$$

Our aim is the study of

$$Inf\ J(\xi) \qquad \xi\in L_2(R^\infty)$$

We can only conjecture, here, that under the hypotheses of Theorem 1, we have

$$\underset{\xi\in L_2(R^\infty)}{Inf}\ J(\xi)=0 \tag{7}$$

We can demonstrate the result in particular case.

THEOREM 2. *We assume that* (5) *and*

$$\pi(t; u, v)=\pi(u, v)$$

are independent of t *then* (7) *is true.*

Outline of Proof

From [7], it is a matter of showing that $u(x, T)$ spans dense in H.

Let us suppose therefore that $\phi\in H$ with

$$(u(T; \xi), \phi)=0 \qquad \forall\xi\in L_2(R^\infty) \tag{8}$$

We introduce the adjoint family $A^*(t)$ of $A(t)$ by (8). Using our hypotheses of theorem 1, there exists a unique $v=v(t)$ such that

$$-\frac{\partial v}{\partial t}+A(t)v=0$$

$$v(t)=\phi$$

$$v\in L_2(0,\ T;\ W_0^1(R^\infty)),\quad v'\in L_2(0,\ T;\ W_0^{-1}(R^\infty))$$

Let us consider the expression

$$\begin{aligned}\int_0^T[(u',\ v)+(u,\ v')]\,dt&=\int_0^T\frac{d}{dt}(u,\ v)dt\\&=(u(T),\ v(T))-(u(0),\ v(0))\\&=-(\xi,\ v(0))\end{aligned}$$

Since $v(T)=\phi$ and $(u(T),\phi)=0$ by hypotheses therefore, $(\xi, v(0))=0,\ \forall\xi\in L_2(R^\infty)$ which $v(0)=0$. But from the backward uniqueness property $v=0$ and hence $\phi=0$ whence the result.

Our Problem

Let (7) be satisfied, with $\eta>0$, given, it is desired to find $\xi_\eta\in L_2(R^\infty)$ such that

$$J(\xi_\eta)\leq\eta \tag{9}$$

In general under the hypotheses of theorem 1, there exists no ξ such that $u_\xi=X$ this is connected with the irreversibility of the problem.

$$\frac{\partial u}{\partial t}+A(t)u=0 \tag{10}$$

$$u=0 \qquad \text{on } \Sigma \tag{11}$$

$$u(x,\ T)=X \tag{12}$$

This is improperly posed.

The Quasi-Reversibility Method

The Q. R. method consists in approximating the systems which are properly posed denoted u_ε for $\varepsilon>0$ the solution of

$$\frac{\partial u_\varepsilon}{\partial t}+A(t)u_\varepsilon-\varepsilon A^*(t)A(t)u_\varepsilon=0 \tag{10}$$

$$u_\varepsilon(T)=X \tag{11}$$

$$u_\varepsilon|_\Sigma=0 \quad\text{and}\quad A(t)u_\varepsilon|_\Sigma=0 \tag{12}$$

$u_\varepsilon \in L_2(0,\ T;\ D(A(t))),\ \dfrac{du}{dt}=u' \in L_2(0,\ T;\ D(A(t))')$

where

$$D(A(t))=\{v:\ v\in W_0'(R^\infty);\ A(t)v\in L_2(R^\infty)\}$$

The domain of the operator $A(t)$ considered as an operator as an unbounded operator in $L_2(R^\infty)$ with the norm

$$\|v\|_{D(A(t))}=(\|v\|^2_{W'_0(R^\infty)}+\|A(t)v\|^2_{L_2(\Omega)})^{1/2}$$

it is a Hilbert space.

$$A^*(t)A(t)\in \mathscr{L}(D(A(t);\ (D(A(t))').$$

THEOREM 3. *Under the hypotheses of theorem* (1) *and suppose that* $DA(t)=V_2$ $=$ *a space independent of* t.

Problem (10-12) *is properly posed, there exists a unique* u_ε *satisfying* (10-12) *and* $u_\varepsilon \in L_2(0,\ T;\ D(A(t))$.

$$u'_\varepsilon \in L_2(0,\ T;\ D(A(t))').$$

PROOF. For $u,\ v\in V_2$, we set

$$a(t,\ u_\varepsilon,\ v_\varepsilon)=(A(t)u_\varepsilon,\ v_\varepsilon)-\varepsilon(A(t)u_\varepsilon,\ A(t)v_\varepsilon)$$

This is a continuous bilinear form and by hypotheses (4) and (5) we have the results

$$-a(t,\ v_\varepsilon,\ v_\varepsilon)+\lambda\|v\|^2_{L_2(R^\infty)}\geq C\|v_\varepsilon\|^2_{V_2} \qquad \forall v_\varepsilon\in V_2,\ C>0$$

Now,

$$\begin{aligned}-a(t,\ v_\varepsilon,\ v_\varepsilon)&=\varepsilon\|A(t)v_\varepsilon\|^2_{L_2(R^\infty)}-(A(t)v_\varepsilon,\ v_\varepsilon)\\ &\geq\varepsilon\|A(t)v_\varepsilon\|^2_{L_2(R^\infty)}-\varepsilon/2\|A(t)v_\varepsilon\|^2_{L_2(R^\infty)}-\frac{1}{2\varepsilon}\|v_\varepsilon\|^2_{L_2(R^\infty)}\end{aligned}$$

with C dependent on ε.

Now, we take $\xi=u_\varepsilon(0)$. Under the given consideration, we may apply the theorems of Lattes and Lions [7] to obtain our result concerning convergence.

THEOREM 4. *Let the hypotheses of theorem* (2), *let* U_ε *be the solution of*

$$\frac{dU_\varepsilon}{dt}+A(t)U_\varepsilon=0$$
$$U_\varepsilon(0)=u_\varepsilon(0)=\xi$$

with u^ε *the solution of* (10-12), *when* $\varepsilon\to 0$ *we have*

$$U_{\varepsilon}(T) \to X.$$

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Mathematics Department, Faculty of Science,
Al-Azhar University, Nasr City, Cairo,
EGYPT