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FRACTIONAL DIFFERENTIAL EQUATIONS

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Abstract: In recent years, several authors have dealt with the fraction derivative [1], in special functions [2], convolution integral equation [5], differintegral equation [4], the derivative of H-function [6], and some other applications. The present paper considers the fraction derivatve in the form of differential eqiation.

1. Introduction

Consider the differential equation of the form

$$\frac{d^\alpha x}{dt^\alpha}=f(t,x(t)),\, t>0,\, 0<\alpha<1 \tag{1.1}$$

Our aim is to examine the concept of solution, its existence, uniqueness, and its coincidence(when $\alpha \longrightarrow 1$) with the one of the initial value problem

$$\frac{dx}{dt}=f(t,x(t)),\quad t>0 \tag{1.2}$$

$$x(0)=0 \tag{1.3}$$

application of singular integral equation is also considered.

2. Properties of Solution

Put $I=(0,T)$, and $D=I\times C(I)$, where $C(I)$ is the class of all continuous functions defined on I, and $||x||=\max_t|x(t)|$, $x\in C(I)$.

LEMMA 2.1. *Let $x\in C(I)$, and $f(t,x(t))\in C(D)$. If a solution of* (1.1) *exists in $C(I)$, then it is given by*

$$x(t)=\frac{1}{\Gamma(\alpha)}\int_0^t \frac{f(\theta,x(\theta))}{(t-\theta)^{1-\alpha}}d\theta \tag{2.1}$$

PROOF. From the properties of the fractional derivatives [1], we can write (1.1) in the form

$$x_{-\alpha}(t)=f(t,x(t))$$

or

$$x(t)*\phi_{-\alpha}=f(t,x(t))$$

taking the convolution with ϕ_α, we get

$$x(t)*\phi_{-\alpha}*\phi_\alpha=f(t,x(t))*\phi_\alpha$$

so

$$x(t)=f(t,x(t))*\phi_\alpha$$
$$=\frac{1}{\Gamma(\alpha)}\int_0^t\frac{f(\theta,x(\theta))}{(t-\theta)^{1-\alpha}}d\theta$$

where

$$\phi_{-k}(t)=\delta^{(k)}(t),\quad k=0,1,2,\cdots$$
$$\phi_{-k}(t)=\frac{t^{-k-1}}{\Gamma_{(-k)}}\qquad \text{for other values of } k.$$

COROLLARY 2.1. *When $\alpha\longrightarrow 1$, the solution* (2.1) *of* (1.1) *coincides with the solution of the initial value problem* (1.2) *and* (1.3).

THEOREM 2.1. *Let $f(t,x(t))\in C(D)$.*

If $f(t,x(t))$ satisfies the Lipschitz condition

$$|f(t,\ x_1(t))-f(t,x_2(t))|\leqq k|x_1(t)-x_2(t)| \tag{2.2}$$

then there exists a unique solution of (1.1) *in $C(I)$, where*

$$T<\frac{\alpha\sqrt{\Gamma(\alpha+1)}}{k}$$

PROOF. Let

$$Fx=\frac{1}{\Gamma(\alpha)}\int_0^t\frac{f(\theta,x(\theta))}{(t-\theta)_{1-\alpha}}d\theta \tag{2.3}$$

Since $f(t,x(t))$ is continuous and satisfies the Lipschitz condition (2.2), it follows that

$$|Fx_1-Fx_2|\leqq\frac{k}{\Gamma_{(\alpha)}}\int_0^t\frac{|x_1(\theta)-x_2(\theta)|}{(t-\theta)^{1-\alpha}}d\theta$$
$$\leqq\frac{k}{\Gamma(\alpha)}\max_t|x_1(t)-x_2(t)|\int_0^t\frac{d\theta}{(t-\theta)^{1-\alpha}}$$

i.e., $$\|Fx_1-Fx_2\|\leqq\frac{k}{\alpha\Gamma(\alpha)}t^\alpha\|x_1-x_2\|,$$

so the map F: $C(I) \longrightarrow C(I)$ is contraction map if

$$\frac{k}{\alpha\Gamma(\alpha)} t^{\alpha} < 1$$

i.e. if
$$t < \sqrt[\alpha]{\frac{\Gamma_{(\alpha+1)}}{k}} = T \tag{2.4}$$

and then by the fixed point theorem [3] we can deduce that there exists a unique solution $x(t) \in C(I)$ of (1.1).

Moreover [3] this solution may be constructed by means of the convergent sequence

$$x_{n+1}(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{f(\theta, x_n(\theta))}{(t-\theta)^{1-\alpha}} d\theta, \ n=1,2,\cdots \tag{2.5}$$

COROLLARY 2.2. *If* $||f(t, x(t))|| \leqq M$, *then*

$$||x|| \leqq \frac{Mt^{\alpha}}{\Gamma(a+1)}, \ t>0 \tag{2.6}$$

PROOF. From (2.1) we can deduce

$$|x(t)| \leqq \frac{1}{\Gamma(\alpha)} \max |f(t, \ x(t))| \int_0^t (t-\theta)^{-1+\alpha} d\theta$$

i.e.
$$||x|| \leqq \frac{M}{\Gamma(\alpha)} \frac{t^{\alpha}}{\alpha} = \frac{Mt^{\alpha}}{\Gamma(\alpha+1)}, \ t>0.$$

3. Application

Now if we want solve the singular integral equation

$$\int_0^t \frac{x(\theta)}{(t-\theta)^{1+\alpha}} d\theta = f(t, x(t)), \ 0<\alpha<1 \tag{3.1}$$

where $x(t) \in C(I), \ I=(0,T)$,

and $f(t, x(t)) \in C(D), \ D=I \times C(I)$.

The properties of the fraction derivatives can be used to reduce (3.1) to the form

$$\frac{1}{\Gamma(-\alpha)} \int_0^t \frac{x(\theta)}{(t-\theta)^{1+\alpha}} d\theta = \frac{1}{\Gamma(-a)} \ f(t, x(t))$$

i.e., to the form

$$\frac{d^{\alpha} x(t)}{dt^{\alpha}} = \frac{1}{\Gamma(-\alpha)} f(t, x(t)) \qquad 0<\alpha<1$$

which by theorem (2.1) has the unique solution given by the sequence

$$x_{n+1}(t)=\frac{1}{\Gamma(-\alpha)\Gamma(\alpha)}\int_0^t\frac{f(\theta,x_n(\theta))}{(t-\theta)^{1-\alpha}}d\theta.$$

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